On Gács' quantum algorithmic entropy

Toru Takisaka

Reserach Institute for Mathematical Sciences, Kyoto University

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background

- Kolmogorov complexity $\cdots$ information content of binary strings

$$C_M(w) = \min\{\, l(v) \mid M(v) = w \,\}.$$

- Approaches to quantize the notion of Kolmogorov complexity
  - Berthiaume et al. (2001)
  - Vitanyi (2001)
  - Gács (2001)
  - etc.

Research object

To make these notions more reliable and more applicable to other research area, relationships between them should be clarified.

background

- Berthiaume et al. (2001) → quantization of Kolmogorov complexity
- Gács (2001) → quantization of lower-semicomputable semimeasure

Theorem (Levin's coding theorem)

$K(x) = -\log \mathbf{m}(x) + O(1).$

→ We expect these quantum notions have some good relationship.

This Talk

It turns out Gács' proof of the following two theorem have some flaw.

Claim

- There is a lower-semicomputable semi-density matrix μ dominating all other such matrices in the sense that for every other such matrix ρ there is a constant $c > 0$ with $\rho \leq c\mu$.
- For a universal operator μ, we define

$$\overline{H}(|\psi\rangle) = -\langle\psi|(\log\mu)\psi\rangle\,,\ \underline{H}(|\psi\rangle) = -\log\langle\psi|\mu\psi\rangle\,.$$

Let $|1\rangle, |2\rangle, \ldots$ be a computable orthogonal sequence of states. Then for $H = \overline{H}$ or $H = \underline{H}$ we have

$$H(|i\rangle) = K(i) + O(1).$$

In this talk, we construct an infinite dimensional modification of Gács' lower-semicomputable semi-density matrices, and discuss their properties and the problem above.

Results

Theorem

Assume a universal operator exists. Then for any uniformly computable ortohomal system $\{\, |\psi_n\rangle \,\}_{n=1}^{\infty}$,

$$K(n) = H(|\psi_n\rangle) + O(1),$$

where $H = \overline{H}$ *or* $H = \underline{H}$*. In particular, for any* $w \in \{0, 1\}^*$,

$$K(w) = H(|w\rangle) + O(1).$$

Theorem

There is no uniformly computable orthonormal basis $\{\, |\psi_n\rangle \,\}_{n=1}^{\infty}$ *of* $\mathcal{H}$ *and lower-semicomputable semimeasure* m *which makes an operator* $\sum_n m(n)|\psi_n\rangle\langle\psi_n|$ *universal.*

We still do not know whether a universal operator exists or not.

Classical notions

Definition

- $f : \{0, 1\}^* \to \mathbb{R}$ *is called* **a semimeasure** *if* $f \geq 0$ *and* $\sum_w f(w) \leq 1$.
- f *is* **lower-semicomputable (LSC)** *if there is a computable function* $\tilde{f} : \{0, 1\}^* \times \mathbb{N} \to \mathbb{Q}$ *such that* $\tilde{f}(w, k) \leq \tilde{f}(w, k + 1)$ *for every* $w \in \{0, 1\}^*$, $k \in \mathbb{N}$, *and* $\tilde{f}(w, k) \xrightarrow{k\to\infty} f(w)$ *for every* w.
- *We call* $\tilde{f}$ *a* **lower-approximation** *of* f.

Theorem

We can enumerate all LSC semimeasures effectively. Namely, there exists $\tilde{m} : \{0, 1\}^* \times \mathbb{N}^2 \to \mathbb{Q}$ *which satisfies following two conditions:*

1. *for any* $n \in \mathbb{N}$, $\tilde{m}(-, -, n)$ *is a lower-approximation of some LSC semimeasure;*
2. *for given LSC semimeasure* m', *there is* $n \in \mathbb{N}$ *such that* $\tilde{m}(-, -, n)$ *is a lower-approximation of* m'.

Classical notions

Theorem

There is a semicomputable semimeasure $\mathbf{m}$ *with the property that for any other semicomputable semimeasure* m' *there is a constant* $c > 0$ *such that for all* x *we have* $cm'(x) \leq \mathbf{m}(x)$.

Proof

$$\mathbf{m}(x) := \sum_{n=1}^{\infty} 2^{-n} m_n(x)$$

is a universal semimeasure, where $\{ m_n \}_{n=1}^{\infty}$ is an effective enumeration of all LSC semimeasures.

Theorem (Levin's coding theorem)

$K(x) = -\log \mathbf{m}(x) + O(1)$.

Notations

- Quantum analogue of binary strings

Classical	Quantum
$\{0,1\}$	$\mathbb{C}^2, \{\ \lvert 0\rangle,\ \lvert 1\rangle\}$
$\{0,1\}^n$	$(\mathbb{C}^2)^{\otimes n}, \{\ \lvert w\rangle\}_{w\in\{0,1\}^n}$
$\{0,1\}^*$	$\bigoplus_{n=0}^{\infty}(\mathbb{C}^2)^{\otimes n}, \{\ \lvert w\rangle\}_{w\in\{0,1\}^*}$

- $\mathcal{H} := \bigoplus_{n=0}^{\infty}(\mathbb{C}^2)^{\otimes n}$ with a computational basis $\{\ \lvert w\rangle\}_{w\in\{0,1\}^*}$
- $\mathcal{L}(\mathcal{H})$: the set of all bounded hermitian operators on $\mathcal{H}$
- $\mathbb{C}_q := \{x+yi \mid x,y\in\mathbb{Q}\}$
- $\langle\psi|\varphi\rangle$: inner product of $|\psi\rangle$ and $|\varphi\rangle$
- $|\psi\rangle\langle\psi|$: one-dimensional projection onto $\mathrm{span}\{\ \lvert\psi\rangle\}$

Definition

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- $\rho \in \mathcal{L}(\mathcal{H})$ *is called a* **semi-density operator** *if* $\rho \geq 0$ *and* $\mathrm{Tr}\rho \leq 1$.
- $\rho \in \mathcal{L}(\mathcal{H})$ *is* **LSC (upper-semicomputable, USC)** *if there is a computable function* $\psi : \mathbb{N} \times \{0,1\}^* \times \{0,1\}^* \to \mathbb{C}_q$ *such that the sequence* $\{\rho_n\}_{n=1}^{\infty} \subset \mathcal{L}(\mathcal{H})$ *defined by*

$$\langle w|\rho_n v\rangle := \psi(n, w, v)$$

satisfies $\rho_n \leq \rho_{n+1}$ ($\rho_n \geq \rho_{n+1}$) *and* $\rho_n \xrightarrow{n\to\infty} \rho$ *in WOT (i.e.* $\langle\psi|\rho_n\psi\rangle \to \langle\psi|\rho\psi\rangle$ *for any* $|\psi\rangle \in \mathcal{H}$*). We call* ψ *a* **lower- (upper-) approximation** *of* ρ.
- $\rho \in \mathcal{L}(\mathcal{H})$ *is* **computable** *if there is a computable function* $\psi : \mathbb{N} \times \{0,1\}^* \times \{0,1\}^* \to \mathbb{C}_q$ *which defines* $\{\rho_n\}_{n=1}^{\infty} \subset \mathcal{L}(\mathcal{H})$ *such that* $\|\rho - \rho_n\| < 2^{-n}$, *in the same manner as above. We call* ψ *an* **approximation** *of* ρ.

Some properties

Question

Is there any relationship between (lower-semi)computability of ρ and $\psi(w, v) = \langle w|\rho v\rangle$?

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- lower-semicomputability of an operator is neither the necessary nor sufficient condition of that of $\psi(w, v)$.

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$f : \{0, 1\}^* \to \mathbb{R}$ is computable if and only if it is LSC and USC. Does it also hold in our quantum version?

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For an eigenvalue decomposition $\sum_n m(n) |\psi_n\rangle \langle\psi_n|$ of ρ, can we find any property of m and $\{ \, |\psi_n\rangle \}_{n=1}^{\infty}$ in a sense of computability?

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Definition

We say a sequence $\{\,|\,\psi_n\rangle\,\}_{n=1}^{\infty}$ *of states is* **uniformly computable** *if there is a recursive function* $\tilde{\psi} : \mathbb{N}^2 \times \{\,0, 1\,\}^* \to \mathbb{C}_q$ *such that*

$$|\langle w|\psi_n\rangle - \tilde{\psi}(k, n, w)| < 2^{-k}F$$

for every $k, n \in \mathbb{N}$ *and* $w \in \{\,0, 1\,\}^*$.

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Question

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Definition

We say a sequence $\{ \, | \, \psi_n \rangle \}_{n=1}^{\infty}$ of states is ***uniformly computable*** *if there is a recursive function $\tilde{\psi} : \mathbb{N}^2 \times \{ 0, 1 \}^* \to \mathbb{C}_q$ such that*

$$|\langle w|\psi_n\rangle - \tilde{\psi}(k, n, w)| < 2^{-k} F$$

for every $k, n \in \mathbb{N}$ and $w \in \{ 0, 1 \}^*$.

Proposition

Let $\{ \, | \, \psi_n \rangle \}_{n=1}^{\infty}$ be a uniformly computable sequence of states, and m be a LSC semimeasure. Then $\rho := \sum_n m(n) |\psi_n\rangle \langle\psi_n|$ is a LSC semi-density operator.

Some properties

- For any LSC semimeasure m, we can take a *positive* approximation function of m.
  - If ψ is a lower-approximation of m then so is $\varphi(x,k) := \max\{\psi(x,k), 0\}$.
- If this property also holds in our quantum extension, then it means we can take a lower-approximation $\{\rho_n\}$ of LSC semi-density operator such that every ρ_n is trace-class, and converges in norm topology.

Proposition

There is a LSC semi-density operator which cannot be approximated by any sequence of positive operators from below.

Some properties

Proof

$$\rho := \left|\frac{1}{2}\lambda + \frac{\sqrt{3}}{2}0\right\rangle\left\langle\frac{1}{2}\lambda + \frac{\sqrt{3}}{2}0\right| = \frac{1}{4}\begin{pmatrix} 1 & \sqrt{3} & 0 & \dots \\ \sqrt{3} & 3 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then ρ is computable, so it is LSC. On the other hand, since ρ is rank-one projection, if there is σ such that $0 \leq \sigma \leq \rho$ then $\sigma = c\rho$ $(0 \leq c \leq 1)$. But it holds that $\langle\lambda|\rho\lambda\rangle \notin \mathbb{C}_q$ or $\langle 0|\rho\lambda\rangle \notin \mathbb{C}_q$ for any $c \in \mathbb{R}\backslash 0$.

Problem

Definition

A LSC semi-density operator μ is **universal** *if for any LSC semi-density operator ν there is $c_\nu > 0$ such that $c_\nu \nu \leq \mu$.*

Problem

Can we enumerate all LSC semi-density operators effectively?

- It cannot be solved just using an analogous approach to the classical situation.

Difficulty of the problem

The way to enumerate all LSC semimeasures

Let $\{ \varphi_n \}_{n=1}^{\infty}$ be an effective enumeration of all partial recursive functions. Consider the following algorithm:

Input $n \in \mathbb{N}$.

1. Let $\alpha_w := 0$ for every $w \in \{ 0, 1 \}^*$.
2. Dovetail φ_n. Whenever φ_n halts for an input $\langle w, k \rangle$, go to step 3.
3. Check whether the conditions $\varphi_n(w, k) \geq \alpha_w$ and $(\sum_{v \neq w} \alpha_v) + \varphi_n(w, k) \leq 1$ hold. If so, then let $\alpha_w := \varphi_n(w, k)$. Otherwise, do nothing. go back to step 2.

Let $\tilde{\psi}(w, t, n)$ be the value of α_w after the t-steps computation of the algorithm above for an input n. Then

- $\tilde{\psi}(-, -, n)$ is an lower-approximation of some LSC semimeasure.
- $\tilde{\psi}$ can approximate any LSC semimeasure from below, since any lower-approximation of a semimeasure is equal to some φ_n, and $\tilde{\psi}(-, -, n)$ approximates the same semimeasure.

Difficulty of the problem

When we naively interpret this proof into the quantum setting, the corresponding algorithm would be like this:
Input $n \in \mathbb{N}$.

1. Let $\alpha_{w,v} := 0$ for every $w, v \in \{0, 1\}^*$.
2. Dovetail φ_n. Whenever φ_n halts for an input $\langle w, v, k \rangle$, go to step 3.
3. Check whether the condition $\varphi_n(w, v, k) \geq \alpha_{w,v}$ holds. If $w = v$, also check whether $(\sum_{u \neq w} \alpha_{u,u}) + \varphi_n(w, v, k) \leq 4$ holds. If so, then let $\alpha_{w,v} := \varphi_n(w, v, k)$. Otherwise, do nothing. go back to step 2.

- The sequence $\{\rho_n\}_{n=1}^{\infty}$ defined in this way is not always increasing.

Difficulty of the problem

We could modify the algorithm as follows:
Input $n \in \mathbb{N}$.

1. Let $\alpha_{w,v} := 0$ for every $w, v \in \{0, 1\}^*$.
2. Dovetail φ_n. Whenever φ_n halts for an input $\langle w, v, k\rangle$, go to step 3.
3. Check whether the condition $\varphi_n(w, v, k) \geq \alpha_{w,v}$ and $\rho_n \leq \rho_{n+1}$ holds. If $w = v$, also check whether $(\sum_{u \neq w} \alpha_{u,u}) + \varphi_n(w, v, k) \leq 4$ holds. If so, then let $\alpha_{w,v} := \varphi_n(w, v, k)$. Otherwise, do nothing. go back to step 2.

- In this modification, generally $\tilde{\psi}(-, -, n)$ does not approximate the same semi-density operator as which is approximated by φ_n from below.

Main results

Proposition

Assume we can enumerate all LSC semi-density operators effectively. Then there is a universal semi-density operator μ.

Proof

$\mu := \sum_{n=1}^{\infty} 2^{-n} \nu_n$ is a universal semi-density operator, where $\{ \nu_n \}_{n=1}^{\infty}$ is an enumeration of LSC semi-density operators.

For a universal operator μ, we define

$$\overline{H}(|\psi\rangle) = -\langle\psi|(\log \mu)\psi\rangle\,,\ \underline{H}(|\psi\rangle) = -\log \langle\psi|\mu\psi\rangle\,.$$

Main results

Proposition

Assume a universal operator exists. Then for any uniformly computable ortohomal system $\{\, |\psi_n\rangle \,\}_{n=1}^{\infty}$,

$$K(n) = H(|\psi_n\rangle) + O(1),$$

where $H = \overline{H}$ *or* $H = \underline{H}$*. In particular, for any* $w \in \{\,0, 1\,\}^*$,

$$K(w) = H(|w\rangle) + O(1).$$

Proof

- The function $f(n) = \langle\psi_n|\mu\psi_n\rangle$ is LSC with $\sum_n f(n) = \mathrm{Tr}\,\mu \leq 1$, hence $K(n) \leq \underline{H}(n) + O(1)$.
- The semi-density operator $\rho = \sum_n \mathbf{m}(n)\,|\psi_n\rangle\,\langle\psi_n|$ is LSC, so

$$K(n) = \langle\psi_n|(-\log\rho)\psi_n\rangle \geq \langle\psi_n|(-\log\mu)\psi_n\rangle + O(1) = \overline{H}(|\psi_n\rangle) + O(1).$$

Main results

Corollary

Assume a universal operator μ exists. Let $\{\,|\,\psi_n\rangle\,\}$ be a uniformly computable orthonormal system (not necessarily a basis) of $\mathcal{H}$. Then a function $\mathbf{m}_\psi(n) := \langle\psi_n|\mu\psi_n\rangle$ is a universal semimeasure.

Main results

Theorem

μ is not diagonal. Namely, There is a LSC semi-density operator which cannot be multiplicatively dominated by $\mu_1 := \sum_i \mathbf{m}(i) |i\rangle \langle i|$.

Proof

Let $\overline{\mathbf{m}}(n) := 2^{-n} \sum_{l(w)=n} \mathbf{m}(w)$. Also let $A_n \in \mathcal{M}_{2^n}(\mathbb{C})$ and $\rho \in \tilde{\mathcal{S}}(\mathcal{H})$ be

$$A_n := \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix}, \rho := \begin{pmatrix} \overline{\mathbf{m}}(0)A_0 & & & \\ & \overline{\mathbf{m}}(1)A_1 & & \\ & & \overline{\mathbf{m}}(2)A_2 & \\ & & & \ddots \end{pmatrix}$$

Then, for $|\psi_n\rangle := 2^{-\frac{n}{2}} \sum_{l(w)=n} |w\rangle$, it holds that

$$\langle \psi_n | \mu_1 \psi_n \rangle = \overline{\mathbf{m}}(n), \langle \psi_n | \rho \psi_n \rangle = 2^n \overline{\mathbf{m}}(n).$$

Hence for any $c > 0$ there is an integer n such that $\langle \psi_n | (\rho - c\mu_1) \psi_n \rangle < 0$.

Main results

Corollary

There is no uniformly computable orthonormal basis $\{\ |\psi_n\rangle\ \}_{n=1}^{\infty}$ of $\mathcal{H}$ and LSC semimeasure m which makes an operator $\sum_n m(n)|\psi_n\rangle\langle\psi_n|$ universal.

Future research

Normal : known facts *Italic* : research objects