(Non-)Reductions in Reverse Mathematics

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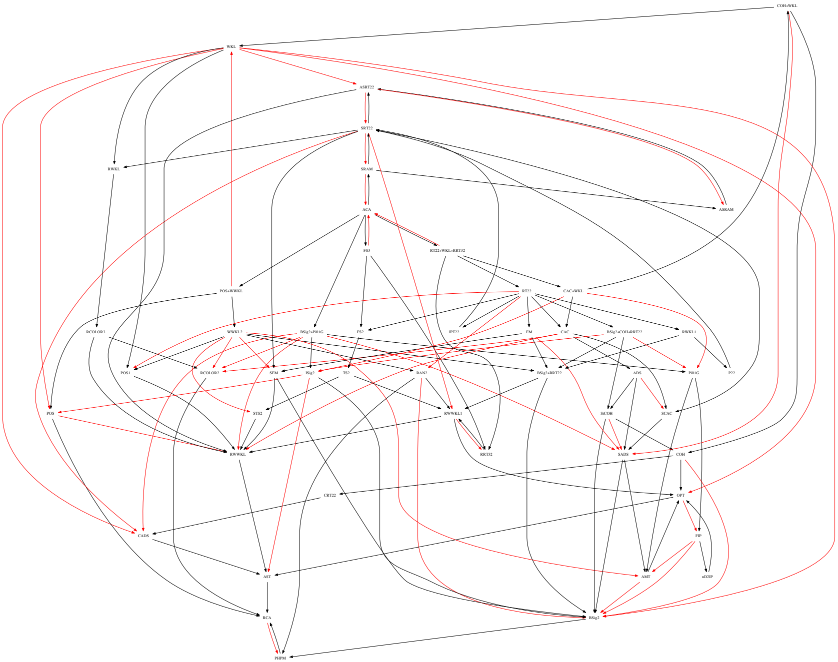
COH+WKL
WKL
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RWWKL1
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SCAC
RWWKL
RRT32
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OPT
CADS
FIP
AST
AMT
nD2IP
RCA
BSig2
PHPM

Much of reverse math focuses on *problems* of the following kind:

> *For every set X there exists a set Y such that $\phi(X, Y)$ holds.*

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Definition (**CAC**)

For every partial order $\sqsubset$ of $\mathbb{N}$ there is either an infinite chain or an infinite antichain.

For every set X there exists a set Y such that $\phi(X, Y)$ holds.

We say that each set X represents an *instance* of the problem, and each witnessing Y is a *solution* to the instance X.

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The reverse mathematics version is:

Does $T + \mathbf{Q} \vdash \mathbf{P}$*?*

T is some base theory. For us it is always $\mathbf{RCA}_0$, which is a theory of computable mathematics.

Theorem (Hirschfeldt/Shore)

$\mathbf{RT}^2_2$ *implies* **ADS** *(working in the base theorem* $\mathbf{RCA}_0$*).*

Proof.

Let $\prec$ be a linear ordering of $\mathbb{N}$. Use Ramsey's Theorem for pairs: define

$$c(n,m) = \begin{cases} 0 & \text{if } \prec \text{ agrees with } < \text{ on } (n,m) \\ 1 & \text{if } \prec \text{ disagrees with } < \text{ on } (n,m) \end{cases}$$

Let H be an infinite set such that c is constant on pairs from H. If c is constantly 0 then listing H in increasing $<$ order gives an infinite ascending sequence. If c is constantly 1 then listing H in increasing $<$ order gives an infinite descending sequence. □

Let $\prec$ be a linear ordering of $\mathbb{N}$. There is a coloring c, computable from $\prec$, so that whenever H is homogeneous for c an infinite monotone sequence for $\prec$ can be computed from H.

For every instance X of **P** there is an instance $\Phi(X)$ of **Q** such that whenever Y is a solution to $\Phi(X)$, $\Psi(X, Y)$ is a solution to X

Definition

We say **P** is *strongly Weihrauch reducible* to **Q** if:
For every instance X of **P** there is an instance $\Phi(X)$ of **Q** such that whenever Y is a solution to $\Phi(X)$, $\Psi(X, Y)$ is a solution to X where Φ, Ψ are computable functionals

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For example, it could be that $\mathbf{RCA}_0 + \mathbf{Q} \vdash \mathbf{P}$ because of the following situation:

> *For every instance X of* **P** *there is an instance $\Phi_0(X)$ of* **Q** *such that whenever Y is a solution to $\Phi_0(X)$ there is an instance $\Phi_1(X, Y)$ of* **Q** *such that whenever Z is a solution to $\Phi_1(X, Y)$, $\Psi(X, Y, Z)$ is a solution to X.*

To say that $\mathbf{P}$ is strongly Weihrauch reducible to $\mathbf{Q}$ is much stronger than saying that $\mathbf{RCA_0} + \mathbf{Q} \vdash \mathbf{P}$.

For example, it could be that $\mathbf{RCA_0} + \mathbf{Q} \vdash \mathbf{P}$ because of the following situation:

> *For every instance X of $\mathbf{P}$ there is an instance $\Phi_0(X)$ of $\mathbf{Q}$ such that whenever Y_0 is a solution to $\Phi_0(X)$ either $\Psi_0(X, Y_0)$ is a solution to X or $\Phi_1(X, Y_0)$ is an instance of $\mathbf{Q}$ such that whenvever Y_1 is a solution to $\Phi_1(X, Y_0)$, either $\Psi_1(X, Y_0, Y_1)$ is a solution to X or $\Phi_2(X, Y_0, Y_1)$ is an instance of $\mathbf{Q}$ such that...*

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Usually the first two steps are the hard part, and iterating is obvious.

We use this to build a model of $\mathbf{RCA}_0 + \mathbf{Q} + \neg\mathbf{P}$:

- Begin by setting H_0 to be the hard instance of $\mathbf{P}$.
- Pick an instance of $\mathbf{Q}$ computable from H_0 and find an easy solution Y_1. Set $H_1 = H_0 \oplus Y_1$.
- Pick an instance of $\mathbf{Q}$ computable from H_1 and find an easy solution Y_2. Set $H_2 = H_1 \oplus Y_2$.
- $\cdots$

Let H consist of all sets computable from some H_n. This is a model $\mathbf{RCA}_0$. By choosing the instances of $\mathbf{Q}$ carefully, we address every instance in H, so H is a model of $\mathbf{Q}$. H was built from "easy" sets, so does not contain a solution to the hard instance of $\mathbf{P}$.

There are various situations where we can prove failure of Weihrauch reducibility, or the stronger failure:

> *For any computable functional Φ there is an instance X of $\mathbf{P}$ such that if $\Phi(X)$ is an instance of $\mathbf{Q}$, there there is a solution Y to $\Phi(X)$ such that $X \oplus Y$ does not compute any solution to X.*

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Note that this is more complicated than the usual diagonalization because the construction of X is intertwined with the construction of Y.

Let $\mathbb{Q}^X_\Phi$ be a forcing notion, given uniformly in oracle X and functional Φ, with a collection of subsets ("requirements") $\mathcal{R}^X_e$. We say $q_1 \succ q_2 \succ \cdots$ is *generic* if for each $\mathcal{R}^X_e$ either:

- Some $q_i \in \mathcal{R}^X_e$, or
- There is some q_i so that whenever $q \preceq q_i$, $q \notin \mathcal{R}^X_e$.

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X will be an instance of **P** and $\mathbb{Q}^X_\Phi$ consists of finite approximations to a solution of $\Phi(X)$, together with Mathias constraints on what future extensions can look like.

Suppose that whenever X is an instance of $\mathbf{P}$ and $q_1 \succ q_2 \succ \cdots$ is a generic sequence in $\mathbb{Q}^X_\Phi$:

- $q_1 \succ q_2 \cdots$ computes a solution Y to $\Phi(X)$, but
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Generic sequences don't always exist. We can't expect X to be just any instance of **P**—there are probably *some* instances of **P** which have computable solutions.

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Then we have shown a non-reducibility result:

> *There exists an instance X of* **P** *and a solution Y to $\Phi(X)$ (computable from $\langle q_i \rangle$) so that $X \oplus \langle q_i \rangle$, and therefore $X \oplus Y$, does not compute a solution to X.*

Let $\mathbb{Q}_\Phi^{X,H}$ be a forcing notion, given uniformly in oracles X and H and a functional Φ, with a collection of subsets ("requirements") $\mathcal{R}_e^{X,H}$. We say $q_1 \succ q_2 \succ \cdots$ is *generic* if for each $\mathcal{R}_e^{X,H}$ either:

- Some $q_i \in \mathcal{R}_e^{X,H}$, or
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- There exists a generic sequence in $\mathbb{Q}_\Psi^{X,H\oplus\langle q_i \rangle}$ for every Ψ.

Suppose further that there exists an X so that $\mathbb{Q}_\Phi^{X,\emptyset}$ contains a generic sequence.

Then we can show $\mathbf{RCA}_0 + \mathbf{Q} \nvdash \mathbf{P}$ as follows:

- Let X be an instance of $\mathbf{P}$ so that $\mathbb{Q}_{\Phi_0}^{X,\emptyset}$ contains a generic sequence. Set $H_0 = X$.

Then we can show $\mathbf{RCA}_0 + \mathbf{Q} \nvdash \mathbf{P}$ as follows:

- Let X be an instance of $\mathbf{P}$ so that $\mathbb{Q}^{X,\emptyset}_{\Phi_0}$ contains a generic sequence. Set $H_0 = X$.
- Given H_n so that $\mathbb{Q}^{X,H_n}_{\Phi_n}$ contains a generic sequence $\langle q_i \rangle$, let $H_{n+1} = H_n \oplus \langle q_i \rangle$.

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We can consider the model of $\mathbf{RCA}_0$ containing all sets computable from some H_n. We can ensure that we consider every instance of $\mathbf{Q}$ at some stage n, so this is a model of $\mathbf{Q}$. This model contains X, but no solution to X, so is a model of $\neg\mathbf{P}$.

- **DNR** vs. **WWKL**
  [Ambos-Spies/Kjos-Hanssen/Lempp/Slaman]

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- Some principles involving partial orders [Dzhafarov/Lerman/Solomon]
- **DNR** vs. **RWKL** [Flood/T.] (shown by Bienvenu/Patey/Shafer using other methods)

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In all known examples the difficult instance of P constructed is computable and can be constructed using a finite injury priority argument.

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Dzhafarov and Lerman/Solomon/T. have shown non-reductions of $\mathbf{SRT}^2_2$ to $\mathbf{RT}^2_2$, but this method appears not to apply. These constructions have an infinite injury character.

Infinite injury creates the following obstacle. We are constructing an instance X of $\mathbf{P}$. We are simultaneously building a generic sequence $q_1 \succ q_2 \succ \cdots$ solving $\Phi_0(X)$, which depends on X. We are also constructing another generic sequence $r_1 \succ r_2 \succ \cdots$ solving $\Phi_1(X, \langle q_i \rangle)$. When the way $\langle q_i \rangle$ depends on X gets too complicated, we lose any control over $\Phi_1(X, \langle q_i \rangle)$, which makes it impossible to ensure that the r_i exist.

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The separation of $\mathbf{SRT}^2_2$ from $\mathbf{RT}^2_2$ by Chong/Slaman/Yang uses a very similar method to construct a collection of solutions. They deal with the infinite injury by only solving instances which are low. This means that the collection of instances which they need to solve doesn't change: they can replace $\Phi_1(X, \langle q_i \rangle)$ with a description that depends only on X.

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Unfortunately, this doesn't work over ω-models.

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The separation of **RWKL** from **DNR** can be shown both by iterated forcing and by a more intrinsic characterization (the "no randomized algorithm" machinery due to Bienvenu/Patay/Shafer).

The end.