Combinatorial Solutions Preserving the Arithmetic Hierarchy

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Basics of Ramsey Theory

$[X]^r$ is the set of r-element subsets of X.

A c-coloring is a function with range contained in $c = \{0, 1, \ldots, c-1\}$.

If a coloring f is constant on $[H]^r$ then H is homogeneous for f.

Theorem (Ramsey)

For every finite r and c, every $f : [\omega]^r \to c$ admits an infinite homogeneous set.

RT^r_c: the instance of Ramsey's Theorem for fixed r, c.

A 2-coloring f of pairs is stable iff $\lim_y f(x, y)$ exists for all x.

SRT^2_2: RT^2_2 for stable 2-colorings of pairs.

A Decomposition of Ramsey's theorem for pairs

ADS: Every infinite linear ordering has an ascending or descending sequence (i.e., a subordering of type ω or ω^* – the reverse ordering of ω).

A 2-coloring of $[\omega]^2$ can be identified as a binary relation on ω (so-called tournament). EM (Erős-Moser) asserts that every tournament R has an infinite set H on which R is transitive (So, R is a linear ordering on H).

Theorem (Bovykin and Weiermann)

$\mathsf{RCA}_0 \vdash \mathsf{RT}^2_2 \leftrightarrow \mathsf{EM} + \mathsf{ADS}$.

Theorem (Hirschfeldt and Shore)

$\mathsf{RCA}_0 + \mathsf{ADS} \nvdash \mathsf{RT}^2_2$.

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An Observation of Jockusch

A stable linear odering is a subordering of $\omega + \omega^*$.

Theorem (Jockusch; Harizanov)

For every (Turing) degree $\mathbf{d} \leq \mathbf{0}'$, there exist $D \in \mathbf{d}$ and a recursive stable linear ordering $<_L$ s.t. D is the ω-part of $<_L$.

Theorem (Hirschfeldt and Shore)

For every recursive stable linear ordering $<_L$, there exists a sequence $S = (a_n : n < \omega)$ s.t. S is of low degree and S is either $<_L$-ascending or $<_L$-descending.

Corollary (Jockusch)

Every degree below the halting problem is of recursively enumerable degree relative to a low degree.

Proof.

If S is an $<_L$-ascending sequence then the ω-part of $<_L$ is recursively enumerable in S. $\square$

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Preserving Properly Δ^0_2 Definitions

A set X preserves (properly) Δ^0_2 definitions (relative to Y) iff every properly Δ^0_2 (Δ^Y_2) set is properly Δ^X_2 ($\Delta^{X\oplus Y}_2$).

Suppose that $\Phi = \forall X \exists Y \varphi(X, Y)$ and φ is arithmetic. Φ admits preservation of Δ^0_2 definitions iff for each X there exists Y s.t. Y preserves Δ^0_2 definitions relative to X and $\varphi(X, Y)$.

SADS: every stable linear ordering admits an infinite ascending or descending sequence.

Corollary

Neither SADS *nor* SRT^2_2 *admits preservation of* Δ^0_2 *definitions.*

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WKL_0

Theorem (Folklore)

Let X and $(A_i : i < \omega)$ be s.t. $A_i \notin \Sigma_1^X$ for all i. Then every non-empty Π_1^X class contains a member G s.t. $A_i \notin \Sigma_1^{X \oplus G}$ for all i.

Thus, WKL_0 admits preservation of Δ_2^0 definitions.

So we have an alternative proof of the following corollary:

Corollary (Hirschfeldt and Shore)

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COH

COH: every sequence $\vec{R} = (R_n : n < \omega)$ of subsets of ω admits a cohesive set C (i.e., C is infinite and for every n either $C \cap R_n$ or $C - R_n$ is finite).

Theorem (WW)

COH *admits preservation of Δ^0_2 definitions.*

A Mathias condition is a pair $(\sigma, X) \in [\omega]^{<\omega} \times [\omega]^{\omega}$ s.t. $\max \sigma < \min X$. We identify (σ, X) with the following set:

$$\{Y : \sigma \subset Y \subseteq \sigma \cup X\}.$$

Lemma

Fix A and (σ, X) with $A \notin \Sigma^X_1$. For every e there exists $(\tau, Y) \subseteq (\sigma, X)$ s.t. $X - Y$ is finite and $A \neq W^Z_e$ for all $Z \in (\tau, Y)$.

Corollary (Hirschfeldt and Shore)

$RCA_0 + WKL_0 + COH \nvdash SADS$.

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Corollary

The Σ^1_1*-theories of* $\mathrm{RCA}_0 + \mathrm{EM}$ *and* $\mathrm{RCA}_0 + \mathrm{SADS}$ *are incomparable.*

E.g., the following Σ^1_1 sentence is a consequence of $\mathrm{RCA}_0 + \mathrm{SADS}$ but not of $\mathrm{RCA}_0 + \mathrm{EM}$:

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A tournament R is stable iff it is induced by a stable 2-coloring of pairs.

SEM: EM for stable tournaments.

With the preservation theorem of COH, the preservation theorem of EM can be reduced to the following preservation lemma of SEM:

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Below we sketch a proof of the above lemma.

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Fix a recursive stable tournament R. Let $f : \omega \to 2$ be as following:

$$f(x) = \begin{cases} 0, & (\forall^\infty y)(xRy); \\ 1, & (\forall^\infty y)(yRx). \end{cases}$$

If H is R-transitive, then $R \upharpoonright [H]^2$ is a stable linear ordering.

For $a \in H$, if $f(a) = 0$ then a belongs to the ω-part, otherwise a belongs to the ω^*-part.

So, $\sigma \in [\omega]^{<\omega}$ can be extended to an infinite R-transitive set, iff σ is R-transitive and

$$aRb \Leftrightarrow f(a) \leq f(b)$$

for all $a, b \in \sigma$ (R and f are compatible on σ).

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Compatibility

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Acceptable Mathias conditions

A Mathias condition (σ, X) is acceptable, iff $\sigma\langle x\rangle$ is R-transitive and R and f are compatible on $\sigma\langle x\rangle$ for all $x \in X$ and

$$(\forall a \in \sigma)(\forall x \in X)((f(a) = 0 \rightarrow aRx) \wedge (f(a) = 1 \rightarrow xRa)).$$

Lemma

If (σ, X) is acceptable then there exists an acceptable $(\tau, Y) \subseteq (\sigma, X)$ s.t. $|\sigma| < |\tau|$ and $X - Y$ is finite.

Proof.

Let $\tau = \sigma\langle x\rangle$ for $x = \min X$.

Let Y be the only infinite set among the following two sets:

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Let $\mathcal{F}$ be the set of $g : \omega \to 2$ s.t. R and g are compatible on $\sigma\langle x\rangle$ for all $x \in X$. So, $\mathcal{F}$ is Π_1^X and $f \in \mathcal{F}$.

Let W be the set of n s.t. for all $g \in \mathcal{F}$ there exists $\xi \in [X]^{<\omega}$ satisfying

- $\sigma\xi$ is R-transitive;
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By the compactness of $\mathcal{F}$, $W \in \Sigma_1^X$ and so $W \neq A_k$. Fix $n \in A_k \triangle W$.

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The key lemma: Case 1

Case 1. $n \in A_k - W$.

Let $\mathcal{U}$ be the set of $g \in \mathcal{F}$ s.t. if $\xi \in [X]^{<\omega}$, $\sigma\xi$ is R-transitive and R and g are compatible on $\sigma\xi$ then $n \notin W_e^{\sigma\xi}$. Then $\mathcal{U} \neq \emptyset$ and $\mathcal{U} \in \Pi_1^X$.

By the preservation of WKL_0, pick $g \in \mathcal{U}$ with $A_i \notin \Sigma_1^{X \oplus g}$ for all i.

Let Y be $X \cap g^{-1}(j)$ s.t. $X \cap g^{-1}(j)$ is infinite.

(σ, Y) is acceptable as (σ, X) is acceptable and $Y \subseteq X$.

As $Y \leq_T X \oplus g$ and $A_i \notin \Sigma_1^{X \oplus g}$, $A_i \notin \Sigma_1^{Y \oplus g}$.

If $Z \in (\sigma, Y)$ then R and g are compatible on Z. So, $n \notin W_e^Z$ for all R-transitive $Z \in (\sigma, Y)$.

So, (σ, Y) is a desirable extension.

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Let $\mathcal{U}$ be the set of $g \in \mathcal{F}$ s.t. if $\xi \in [X]^{<\omega}$, $\sigma\xi$ is R-transitive and R and g are compatible on $\sigma\xi$ then $n \notin W_e^{\sigma\xi}$. Then $\mathcal{U} \neq \emptyset$ and $\mathcal{U} \in \Pi_1^X$.

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(σ, Y) is acceptable as (σ, X) is acceptable and $Y \subseteq X$.

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SEM

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Case 2. $n \in W - A$.

Fix $\xi \in [X]^{<\omega}$ s.t. $\sigma\xi$ is R-transitive, R and f are compatible on $\sigma\xi$ and $n \in W_e^{\sigma\xi}$.

Let $\tau = \sigma\xi$. As τ is R-transitive, it can be listed in R-ascending order:

$$a_0 R a_1 R \ldots R a_{k-1}, k = |\tau|.$$

Let

$$X_0 = \{x \in X : x > \max\tau, x R a_0\},$$
$$X_i = \{x \in X : x > \max\tau, a_{i-1} R x R a_i\} (0 < i < k),$$
$$X_k = \{x \in X : x > \max\tau, a_{k-1} R x\}.$$

Let Y be the unique infinite X_i. Then (τ, Y) is as desired.

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FS^2

A set H is free for $f : [\omega]^r \to \omega$ iff $f(\sigma) \notin H - \sigma$ for all $\sigma \in [H]^r$; a set G is thin for f iff $f([G]^r) \neq \omega$.

FS^r (TS^r): every $f : [\omega]^r \to \omega$ admits an infinite free (thin) set.

Theorem (H. Friedman; Cholak, Giusto, Hirst and Jockusch)

$\mathsf{RCA}_0 \vdash \mathsf{RT}^r_2 \to \mathsf{FS}^r_2 \to \mathsf{TS}^r_2$

Theorem (WW)

Every recursive $f : [\omega]^2 \to \omega$ admits an infinite free set preserving Δ^0_2 definitions.

Theorem (WW)

(RCA_0) The Σ^1_1-theories of FS^2 (TS^2) and SADS are incomparable. Thus FS^2 (TS^2) is strictly weaker than RT^2_2.

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Free Sets for Arbitrary Functions

To prove the preservation theorem for FS^2, it suffices to combine the preservation theorem for cohesive sets and the following theorem.

Theorem

Every $f : \omega \to \omega$ admits an infinite free set preserving Δ^0_2 definitions.

The above theorem can be reduced to the following:

Lemma

1. *If X is Martin-Löf random relative to $f : \omega \to \omega$ s.t. $f(x) \geq x$ for all x, then X computes an infinite free set for f;*
2. *Every $f : \omega \to \omega$ s.t. $f(x) \leq x$ for all x admits an infinite free set preserving Δ^0_2 definitions.*

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Free Sets for Arbitrary Regressive Functions

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Proof.

If there exists an infinite X s.t. X preserves Δ^0_2 definitions and $f(X)$ is finite, then $X - b$ is f-free for some b.

Suppose that there is no such X. If (σ, X) is a Mathias condition s.t. σ is f-free and X preserves Δ^0_2 definitions, then σ can be extended to an infinite f-free $Y \in (\sigma, X)$. With this simple but useful observation, we can build a free set, by forcing with conditions (σ_0, σ_1, X) s.t.

1. (σ_i, X) is a Mathias condition;
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Climbing up the Arithmetic Hierarchy

Preserving the arithmetic hierarchy

A set X preserves (properly) Ξ-definitions (relative to Y) for Ξ among $\Delta^0_{n+1}, \Pi^0_n, \Sigma^0_n$ where $n > 0$, iff every properly Ξ (Ξ^Y) set is properly Ξ^X ($\Xi^{X \oplus Y}$).

X preserves the arithmetic hierarchy (relative to Y) iff X preserves Ξ-definitions (relative to Y) for all Ξ among $\Delta^0_{n+1}, \Pi^0_n, \Sigma^0_n$ where $n > 0$.

Proposition (Folklore)

If G is sufficiently Cohen generic (Martin-Löf random) then G preserves the arithmetic hierarchy.

Suppose that $\Phi = \forall X \exists Y \varphi(X, Y)$ and φ is arithmetic. Φ admits preservation of the arithmetic hierarchy iff for each X there exists Y s.t. Y preserves the arithmetic hierarchy relative to X and $\varphi(X, Y)$.

Corollary

These statements admit preservation of the arithmetic hierarchy: RRT^2_2, WWKL_0, $\Pi^0_1\mathrm{G}$, AMT, OPT.

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WKL_0 *admits preservation of the arithmetic hierarchy.*

Proof.

Let T be a recursive infinite binary tree. We build a desired $G \in [T]$ by forcing with primitively recursive subtrees of T: $S \in \mathbb{P}$ iff S is an infinite binary tree of the following form

$$S = T \cap R$$

where R is a primitively recursive subset of $2^{<\omega}$.

We define $S \Vdash \varphi$ for arithmetic φ as usual. For $n > 0$, it can be shown that $S \Vdash \varphi$ is Σ^0_n (Π^0_n) definable if φ is a Σ^0_n (Π^0_n) sentence. □

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Climbing up the Arithmetic Hierarchy

RT^2_2

We know that RT^2_2 does not admit preservation of Δ^0_2 definitions as it is stronger than SADS.

Theorem (WW)

RT^2_2 admits preservation of Ξ definitions for Ξ among $\Sigma^0_{n+1}, \Pi^0_{n+1}, \Delta^0_{n+2}$ where $n > 0$.

Proof.

By relativizing the last preservation theorem of WKL_0, we get P s.t. P is PA over $\emptyset'$ and every properly $\Xi^{\emptyset'}$ set is properly Ξ^P for Ξ among $\Sigma^0_n, \Pi^0_n, \Delta^0_{n+1}$ where $n > 0$.

By a theorem of Cholak, Jockusch and Slaman, every recursive 2-coloring of pairs admits an infinite homogeneous set H with $H' \leq_T P$. Hence for $n > 0$ and Ξ among $\Sigma^0_{n+1}, \Pi^0_{n+1}, \Delta^0_{n+2}$, every properly Ξ set is properly Ξ^H. $\square$

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By a theorem of Cholak, Jockusch and Slaman, every recursive 2-coloring of pairs admits an infinite homogeneous set H with $H' \leq_T P$. Hence for $n > 0$ and Ξ among $\Sigma^0_{n+1}, \Pi^0_{n+1}, \Delta^0_{n+2}$, every properly Ξ set is properly Ξ^H. □

We know that RT^2_2 does not admit preservation of Δ^0_2 definitions as it is stronger than SADS.

Theorem (WW)

RT^2_2 admits preservation of Ξ definitions for Ξ among $\Sigma^0_{n+1}, \Pi^0_{n+1}, \Delta^0_{n+2}$ where $n > 0$.

Proof.

By relativizing the last preservation theorem of WKL_0, we get P s.t. P is PA over $\emptyset'$ and every properly $\Xi^{\emptyset'}$ set is properly Ξ^P for Ξ among $\Sigma^0_n, \Pi^0_n, \Delta^0_{n+1}$ where $n > 0$.

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Questions

1. Are there other combinatorial principles which admit preservation of the arithmetic hierarchy? E.g., does every uniformly recursive $(R_n : n < \omega)$ admit a cohesive set which preserves the arithmetic hierarchy?
2. How can we exploit such preservation? Does it lead to any deeper metamathematical consequences?

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Thanks!