Isolation: with applications considered

Wu Guohua

School of Physical and Mathematical Sciences

Nanyang Technological University

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Isolation: a short history

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Definition

A d.c.e. degree $\mathbf{d}$ is isolated by a c.e. degree $\mathbf{a}$, if $\mathbf{a} < \mathbf{d}$ and all c.e. degrees below $\mathbf{d}$ are also below $\mathbf{a}$, or, equivalently, $\mathbf{a}$ is the greatest c.e. degree below $\mathbf{d}$.

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This definition was proposed by Cooper and Yi in a preprint in 1995.

- Cooper and Yi had a direct construction of isolation pairs. They also proved that not every nonzero d.c.e. degrees are isoated.
- Ding and Qian, and LaForte, independently, proved that isolated degrees exists between any two c.e. degrees.
- Arslanov, Lempp and Shore proved that nonisolated degrees exists between any two c.e. degrees. They also proved that nonisolating degrees exist (downward density).
- Salts proved that nonisolating degrees are not dense in the c.e. degrees, while the upward density is true.

Lachlan Sets and Ishmukhametov's exact degrees

Lachlan Sets

For a d.c.e. set X with a d.c.e enumeration $\{X_s\}_{x \in \mathbb{N}}$, the corresponding Lachlan set $L(X)$ is defined as

$$\{s \mid \exists x[x \in X_s - X_{s-1} \text{ and } x \notin X]\}.$$

Obviously, $L(X)$ is c.e., $L(X) \leq_T X$ and X is c.e. in $L(X)$.
For a d.c.e. degree $\mathbf{d} > \mathbf{0}$, we denote by $L[\mathbf{d}]$ the class of degrees of Lachlan sets of those d.c.e. sets in $\mathbf{d}$.

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Ishmukhametov proved that $L[\mathbf{d}]$ can be a singleton, in which case $\mathbf{d}$ is called a exact degree, and that $L[\mathbf{d}]$ can be an interval. Obviously, exact degrees are isolated degrees.

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Fang, Wu and Yamaleev proved that $L[\mathbf{d}]$ can have no minimal elements.

Downey's Diamond Theorem

There are incomparable d.c.e. degrees $\mathbf{d}_1, \mathbf{d}_2$ such that $\mathbf{d}_1 \cup \mathbf{d}_2 = \mathbf{0}'$ and $\mathbf{d}_1 \cap \mathbf{d}_2 = \mathbf{0}$.
Thus $\{\mathbf{0}, \mathbf{d}_1, \mathbf{d}_2, \mathbf{0}'\}$ is a diamond embedding.

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Diamond Embedding via Isolation

There are a d.c.e. degree **d** and c.e. degrees **a**, **c** such that

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Further Improvement

For any nonzero c.e. degree $\mathbf{c}$, if $\mathbf{c}$ is cappable, then there are a d.c.e. degree $\mathbf{d}$ and c.e. degrees $\mathbf{a}$¡ $\mathbf{d}$ such that

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It implies that any cappable c.e. degree is complemented in the d.c.e. degrees.

Almost Universal Cupping degrees

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- Compared to the construction of maximal d.r.e. degrees, the construction of d.c.e. degrees with (only) almost universal cupping property is much easier.
- Liu and Wu constructed an isolated d.c.e. degree $\mathbf{d}$ with the almost cupping property.

Requirements

The constructed sets need to satisfy the isolation requirements and also the following cupping requirements:

$\mathcal{R}_e$: $K = \Gamma_e^{B,D,W_e} \;\vee\; W_e = \Delta_e^B$, where Γ_e and Δ_e are p.c. functionals constructed by us.

Theorem (Fang, Liu and Wu)

Fang, Liu and Wu proved recently that for any nonzero cappable c.e. degree **c**, there is a d.c.e. degree **d** with almost universal cupping property and a c.e. degree $\mathbf{b} < \mathbf{d}$ such that

- **b** isolates **d**, and
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1. By applying this result twice, first to $\mathbf{c}$ and then to $\mathbf{b}$ — we have $\mathbf{d}$ and $\mathbf{b}$ first, and then $\mathbf{e}$ and $\mathbf{a}$ such that $\mathbf{e}$ has almost universal cupping property and $\mathbf{a} < \mathbf{e}$ isolates $\mathbf{e}$, and $\mathbf{a}$ and $\mathbf{b}$ form a minimal pair.

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3. Obviously, this result has Li-Yi's cupping theorem as a direct corollary.

Thanks!