Interpretations into Weihrauch lattice

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Background & Objective

Background

Weihrauch lattice

- A degree structure from computable analysis
- Its underlying reducibility requires uniform computability
- Medvedev lattice is embeddable into Weihrauch lattice

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BGM-Program (V. Brattka, G. Gherardi, A. Marcone)

- To classify non-constructive principles in Weihrauch lattice
- Each non-constructive principle is "impemented" as a Weihrauch degree (Not in an automatic way)
- Not a formal approach of logic (at least for pure logician)
- Unknown relationship to CRM :≡ Constructive Reverse Math.

Objective

To find a formal connection between CRM and BGM-program

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Tools

1. Cut-Elimination Theorem (Proof Theory)

   Any derivable sequent is derivable without cut

2. Equivalence Theorem (Categorical Logic)

   Each model is equivalent to the term model of its internal theory

Cut-Elimination Theorem

SIL: 1/6

Typed Lambda Calculus ($\lambda_{(\times,\to)}$-calculus)

Signature language + axioms for typing judgment ($\Gamma \vdash t : \sigma$)
(i.e. type assignment for function symbols)

Specification signature + axioms for conversion judgment ($\Gamma \vdash t = u : \sigma$)

meta variables

- $x, y, \cdots$ for variables
- $\alpha, \beta, \cdots$ for base types
- $f, g, \cdots$ for function symbols
- $\sigma, \tau, \cdots$ for types

  $$\sigma ::= 1 \mid \alpha \mid \sigma \to \sigma \mid \sigma \times \sigma$$
- $t, u, \cdots$ for terms

  $$t ::= \langle\rangle \mid f(t, \cdots, t) \mid \lambda x{:}\sigma.t \mid t(t) \mid \langle t, t\rangle \mid \pi t \mid \pi' t$$
- $\Gamma, \Delta, \cdots$ for type contexts, Λ for the empty context

  $$\Gamma \equiv x_1 : \sigma_1, \cdots, x_k : \sigma_k \qquad (x_1, \cdots, x_k : \text{distinct})$$

SIL: 2/6

Extention of Language

Logical Constants : $\bot$

Predicate Symbols : $=, P \in \Pi_p$ (Π_p: given)

Logical Connectives: $\wedge, \vee, \rightarrow$

Quantifiers : $\forall, \exists$

Signature for SIL

A specification for typed lambda calculus equipped with a mapping

$$S' : P \mapsto (\sigma_1, \cdots, \sigma_k) \; ({}^{\forall}P \in \Pi_p)$$

$P(-, \cdots, -)$: finite symbol sequence with holes as $|S' f|$

Formula

(meta variables: $\varphi, \psi, \chi, \cdots$)

$$\varphi ::= \bot \mid P(t, \cdots, t) \mid t =_\sigma t \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \exists x : \sigma.\varphi \mid \forall x : \sigma.\varphi$$

Expressions

Typing Judgment: $\Gamma \vdash \varphi : \mathtt{Prop}$

Sequent : $\Gamma \mid \Theta \vdash \varphi$ ($\Theta, \Xi, \cdots$ for finite sequences of formulae)

abbreviation: $t^\sigma_\Gamma :\equiv (\Gamma, \sigma, t)$ if $\Gamma \vdash t : \sigma$ is derivable
$\varphi_\Gamma :\equiv (\Gamma, \varphi)$ if $\Gamma \vdash \varphi : \mathtt{Prop}$ is derivable

Specification for SIL

A signature equipped with a set $\mathscr{A}$ of sequents (axiom set) closed under:

$$\frac{\Gamma_0, x_0 : \sigma_0, x_1 : \sigma_1, \Gamma_1 \mid \Theta \vdash \varphi}{\Gamma_0, x_1 : \sigma_1, x_0 : \sigma_0, \Gamma_1 \mid \Theta \vdash \varphi}\,(\mathrm{E})_t \qquad \frac{\Gamma \mid \Theta \vdash \varphi}{\Gamma, x : \sigma \mid \Theta \vdash \varphi}\,(\mathrm{W})_t$$

$$\frac{\Gamma, x_0 : \sigma, x_1 : \sigma \mid \Theta \vdash \varphi}{\Gamma, x_0 : \sigma \mid \Theta[x_0/x_1] \vdash \varphi[x_0/x_1]}\,(\mathrm{C})_t \qquad \frac{\Gamma \vdash t : \sigma \qquad \Gamma, x : \sigma \mid \Theta \vdash \varphi}{\Gamma \mid \Theta[t/x] \vdash \varphi[t/x]}\,(\mathrm{S})$$

We denote by SIL the specification whose axiom set is empty

SIL: 4/6

$$\frac{\Gamma \mid \Theta_0, \psi_0, \psi_1, \Theta_1 \vdash \varphi}{\Gamma \mid \Theta_0, \psi_1, \psi_0, \Theta_1 \vdash \varphi}\,(\mathrm{E}) \qquad \frac{\Gamma \mid \Theta, \psi, \psi \vdash \varphi}{\Gamma \mid \Theta, \psi \vdash \varphi}\,(\mathrm{C})$$

$$\frac{\Gamma \mid \Theta \vdash \psi \qquad \Gamma \mid \Xi, \psi \vdash \varphi}{\Gamma \mid \Theta, \Xi \vdash \varphi}\,(\mathrm{Cut})$$

"ψ" in (Cut) is called *cut-formula*

$$\frac{\Gamma \vdash \Theta, \varphi : \mathtt{Prop}}{\Gamma \mid \Theta, \varphi \vdash \varphi}\,(\mathrm{Id}) \qquad \frac{\Gamma \vdash \Theta, \varphi : \mathtt{Prop}}{\Gamma \mid \Theta, \bot \vdash \varphi}\,(\bot) \qquad \frac{(\Gamma \mid \Theta \vdash \varphi) \in \mathscr{A}}{\Gamma \mid \Theta \vdash \varphi}\,(\mathscr{A})$$

$$\frac{\Gamma \vdash \Theta : \mathtt{Prop} \qquad \Gamma \vdash t = u : \sigma}{\Gamma \mid \Theta \vdash t =_\sigma u}\,(\mathrm{Eq})$$

$$\frac{\Gamma \vdash \Theta : \mathtt{Prop} \qquad \Gamma \vdash t_0, t_1 : \sigma \qquad \Gamma, x : \sigma \vdash \varphi : \mathtt{Prop}}{\Gamma \mid \Theta, t_0 =_\sigma t_1, \varphi[t_i/x] \vdash \varphi[t_{1-i}/x]}\,(\mathrm{R})$$

SIL: 5/6

$$\frac{\Gamma \mid \Theta, \psi_0, \psi_1 \vdash \varphi}{\Gamma \mid \Theta, \psi_0 \wedge \psi_1 \vdash \varphi}\ (\wedge\mathrm{L}) \qquad \frac{\Gamma \mid \Theta \vdash \varphi_0 \qquad \Gamma \mid \Theta \vdash \varphi_1}{\Gamma \mid \Theta \vdash \varphi_0 \wedge \varphi_1}\ (\wedge\mathrm{R})$$

$$\frac{\Gamma \mid \Theta, \psi_0 \vdash \varphi \qquad \Gamma \mid \Theta, \psi_1 \vdash \varphi}{\Gamma \mid \Theta, \psi_0 \vee \psi_1 \vdash \varphi}\ (\vee\mathrm{L}) \qquad \frac{\Gamma \mid \Theta \vdash \varphi_i \qquad \Gamma \vdash \varphi_{1-i} : \mathtt{Prop}}{\Gamma \mid \Theta \vdash \varphi_0 \vee \varphi_1}\ (\vee\mathrm{R})$$

$$\frac{\Gamma \mid \Theta \vdash \psi_0 \qquad \Gamma \mid \Theta, \psi_1 \vdash \varphi}{\Gamma \mid \Theta, \psi_0 \to \psi_1 \vdash \varphi}\ (\to\mathrm{L}) \qquad \frac{\Gamma \mid \Theta, \varphi_0 \vdash \varphi_1}{\Gamma \mid \Theta \vdash \varphi_0 \to \varphi_1}\ (\to\mathrm{R})$$

SIL: 6/6

$$\frac{\Gamma \vdash \Theta, \exists x{:}\sigma.\psi, \varphi : \mathtt{Prop} \qquad \Gamma, y{:}\sigma \mid \Theta, \psi[y/x] \vdash \varphi}{\Gamma \mid \Theta, \exists x{:}\sigma.\psi \vdash \varphi} \ (\exists L)$$

$$\frac{\Gamma \vdash t{:}\sigma \qquad \Gamma \mid \Theta \vdash \varphi[t/x]}{\Gamma \mid \Theta \vdash \exists x{:}\sigma.\varphi} \ (\exists R)$$

$$\frac{\Gamma \vdash t{:}\sigma \qquad \Gamma \mid \Theta, \varphi[t/x] \vdash \psi}{\Gamma \mid \Theta, \forall x{:}\sigma.\varphi \vdash \psi} \ (\forall L)$$

$$\frac{\Gamma \vdash \Theta, \forall x{:}\sigma.\varphi : \mathtt{Prop} \qquad \Gamma, y{:}\sigma \mid \Theta \vdash \varphi[y/x]}{\Gamma \mid \Theta \vdash \forall x{:}\sigma.\varphi} \ (\forall R)$$

"y" in (∃L) and (∀R) is called *temporal variable*

Cut-Elimination Theorem: 1/5

Pure Variable Derivation

A derivation $\mathscr{D}$ with the following conditions:

- $\mathrm{BV}(\mathscr{D}) \cap \mathrm{FV}(\mathscr{D})$
- for each occurence of temporal variable, the variable is not used as temporal variable in other places

Fact

Given a derivation, it can be transformed into a pure variable derivation by renaming bound variables and temporal variables

Cut-Elimination Theorem

Given a pure variable derivation over SIL, one finds an essential-cut free derivation of the same conclusion

an *essential-cut* :≡ a cut whose cut-formula is not atomic

Cut-Elimination Theorem: 2/5

ρ: fixed type

Parametrization

Let $\psi :\equiv \forall y{:}\tau.\psi_0$. Define:

$$^{\rho}\psi \quad :\equiv \quad \forall y{:}(\rho \to \tau).\forall z{:}\rho.\psi_0[yz/y]$$

(z:the first flesh variable symbol)

$\Gamma \mid \psi_\Gamma \vdash {}^{\rho}\psi_\Gamma$ is derivable over SIL, and the converse one also derivable whenever ρ is "inhabitant" relative to Γ

Idempotency

ρ is *idempotent* iff ρ is "isomorphic" to $\rho \times \rho$
i.e. there is a pair t and u of terms s.t. the following judgments are derivable

- $\Lambda \vdash tu = (\lambda x{:}\rho.x){:}\rho \to \rho$
- $\Lambda \vdash ut = (\lambda x{:}\rho \times \rho.x){:}(\rho \times \rho) \to (\rho \times \rho)$

Cut-Elimination Theorem: 3/5

Notation

$$
\begin{array}{rcll}
\mathrm{Sub}^+(a) & = & \{a\} & (a : atomic) \\
\mathrm{Sub}^+(\varphi_0 \vee \varphi_1) & = & \mathrm{Sub}^+(\varphi_0) \cup \mathrm{Sub}^+(\varphi_1) \cup \{\varphi_0 \vee \varphi_1\} & \\
\mathrm{Sub}^+(\varphi_0 \wedge \varphi_1) & = & \mathrm{Sub}^+(\varphi_0) \cup \mathrm{Sub}^+(\varphi_1) \cup \{\varphi_0 \wedge \varphi_1\} & \\
\mathrm{Sub}^+(\varphi_0 \rightarrow \varphi_1) & = & \mathrm{Sub}^-(\varphi_0) \cup \mathrm{Sub}^+(\varphi_1) \cup \{\varphi_0 \rightarrow \varphi_1\} & \\
\mathrm{Sub}^+(\exists x : \sigma.\varphi_0) & = & \mathrm{Sub}^+(\varphi_0) \cup \{\exists x : \sigma.\varphi_0\} & \\
\mathrm{Sub}^+(\forall x : \sigma.\varphi_0) & = & \mathrm{Sub}^+(\varphi_0) \cup \{\forall x : \sigma.\varphi_0\} & \\
\mathrm{Sub}^-(a) & = & \emptyset & (a : atomic) \\
\mathrm{Sub}^-(\varphi_0 \vee \varphi_1) & = & \mathrm{Sub}^-(\varphi_0) \cup \mathrm{Sub}^-(\varphi_1) & \\
\mathrm{Sub}^-(\varphi_0 \wedge \varphi_1) & = & \mathrm{Sub}^-(\varphi_0) \cup \mathrm{Sub}^-(\varphi_1) & \\
\mathrm{Sub}^-(\varphi_0 \rightarrow \varphi_1) & = & \mathrm{Sub}^+(\varphi_0) \cup \mathrm{Sub}^-(\varphi_1) & \\
\mathrm{Sub}^-(\exists x : \sigma.\varphi_0) & = & \mathrm{Sub}^-(\varphi_0) & \\
\mathrm{Sub}^-(\forall x : \sigma.\varphi_0) & = & \mathrm{Sub}^-(\varphi_0) &
\end{array}
$$

Cut-Elimination Theorem: 4/5

Positive Universal Quantification Free Relative to ρ

- A formula φ is ${}^{\rho}$*p.u.f.*
  iff no formula of the form $\forall x\!:\!\sigma.\psi_0$ $(\sigma \not\equiv \rho)$ belongs to $\mathrm{Sub}^{+}(\varphi)$
- An axiom set $\mathscr{A}$ is ${}^{\rho}$*p.u.f.*
  iff $(\bigwedge\!\!\!\bigwedge \Theta) \to \varphi$ is ${}^{\rho}$p.u.f. for each $(\Gamma \mid \Theta \vdash \varphi) \in \mathscr{A}$
- A specification is ${}^{\rho}$*p.u.f.* iff its axiom set is ${}^{\rho}$p.u.f.

According usage:

${}^{\rho}$p.e.f. (positive existencial quantificatioin free relative to ρ),
${}^{\rho}$n.u.f. (negative universal quantificatioin free relative to ρ),
${}^{\rho}$n.e.f. (negative existencial quantificatioin free relative to ρ),
${}^{\rho}$q.f. (quantification free relative to ρ),...

Cut-Elimination Theorem: 5/5

Lemma 1 < Cut-Elimination Theorem

Assume that:

- $\mathscr{A}$ is ${}^{\rho}$p.e.f. and ${}^{\rho}$n.u.f.
- ψ_0 is ${}^{\rho}$p.e.f. and ${}^{\rho}$n.u.f.
- φ is ${}^{\rho}$p.u.f. and ${}^{\rho}$n.e.f.
- ρ is idempotent

If $\Gamma \mid \forall y\!:\!\tau.\psi_0 \vdash \varphi$ is derivable over $\mathscr{A}$, there is a finite sequence $t_{1}{}^{\rho\to\tau}_{\Gamma}, \cdots, t_{k}{}^{\rho\to\tau}_{\Gamma}$ of terms s.t.

$$\Gamma \mid (\forall z\!:\!\rho.\psi_0[yz/y])[t_{1}{}^{\rho\to\tau}_{\Gamma}/y], \cdots, (\forall z\!:\!\rho.\psi_0[yz/y])[t_{k}{}^{\rho\to\tau}_{\Gamma}/y] \vdash \varphi$$

is derivable over $\mathscr{A}$

Remark

From the resulting "witnessed" sequent, $\Gamma \mid {}^{\rho}(\forall y\!:\!\tau.\psi_0) \vdash \varphi$ is derivable

Equivalence Theorem

First Order Fibration: 1/5

First order fibration

- A kind of functor
- A categorical abstraction of "predicate logic"
- Correspondence:

Logic	Fibration $p : \mathbb{E} \to \mathbb{B}$
type	object of $\mathbb{B}$
term	morphism in $\mathbb{B}$
formula	object of $\mathbb{E}$

Construction

Logic-to-Fibration: term model

Fibration-to-Logic: internal theory

First Order Fibration: 2/5

Term Model

Fix a specification $\mathscr{A}$ for SIL

- $\lambda(\mathscr{A})$

  obj. : types

  morph. : $\sigma \xrightarrow{[t]_\sim} \tau$ in $\lambda(\mathscr{A})$

  ($t \sim u$ iff $v_0 : \sigma \vdash t = u : \tau$ is derivable)

- $\mathcal{L}(\mathscr{A})$

  obj. : $\varphi_{v_0:\sigma}$

  morph. : $\varphi_{v_0:\sigma} \xrightarrow{[t]_\sim} \psi_{v_0:\tau}$ in $\mathcal{L}(\mathscr{A})$

  $\iff v_0 : \sigma \mid \varphi \vdash \psi[t/v_0]$ is derivable

- $\mathcal{T}(\mathscr{A}) : \mathcal{L}(\mathscr{A}) \to \lambda(\mathscr{A})$

  $\varphi_{v_0:\sigma} \mapsto \sigma$

  $[t]_\sim \mapsto [t]_\sim$

$\mathcal{T}(\mathscr{A})$ is a first order fibration with Cartesian closed base category

First Order Fibration: 3/5

Internal Theory

Fix a first order fibration $p : \mathbb{E} \to \mathbb{B}$ with Cartesian closed base category.

Base Type Sym.: Names for obj. of $\mathbb{B}$: $A \mapsto \overline{A}$

$$[\![1]\!] := 1, [\![\overline{A}]\!] := A, [\![\sigma_0 \times \sigma_1]\!] := [\![\sigma_0]\!] \times [\![\sigma_1]\!],$$
$$[\![\sigma_0 \to \sigma_1]\!] := [\![\sigma_0]\!] \to [\![\sigma_1]\!]$$

Func. Sym. : Names for morph. in $\mathbb{B}$: $([\![\sigma_0]\!] \xrightarrow{f} [\![\sigma_1]\!]) \mapsto (\overline{f} : \sigma_0 \to \sigma_1)$

$$(1 \xrightarrow{c} [\![\sigma]\!]) \quad \mapsto \quad (\overline{c} :\to \sigma)$$

$$[\![\langle\rangle]\!]_{v_0:\sigma} := ([\![\sigma]\!] \xrightarrow{!} 1), [\![\overline{f}(t)]\!]_{v_0:\sigma_0} := f \circ [\![t]\!]_{v_0:\sigma},$$
$$[\![\langle t_0, t_1\rangle]\!]_{v_0:\sigma} := \langle [\![t_0]\!], [\![t_1]\!]\rangle, \cdots \text{ (Omit)}$$

Axiom(Conv.) : Add $v_0 : \sigma \vdash t_0 = t_1 : \tau$ iff $[\![t_0]\!]_{v_0:\sigma} = [\![t_1]\!]_{v_0:\sigma}$

Predic. Sym. : Names for obj. of $\mathbb{E}$: $X \mapsto \overline{X}$

$$[\![\overline{X}]\!] := X, \cdots \text{ (Omit)}$$

Axiom(Seq.) : Add $v_0 : \sigma \mid \psi \vdash \varphi$ iff there is a morphism

$$[\![\psi]\!]_{v_0:\sigma} \xrightarrow{j} [\![\varphi]\!]_{v_0:\sigma} \text{ in } \mathbb{E} \text{ s.t. } pj = \mathrm{id}_{[\![\sigma]\!]}$$

First Order Fibration: 4/5

Notation

$\mathrm{Spec}_{\mathrm{SIL}}$: cat. of specifications
(morphism: translations)

$\mathrm{Fib}_{\mathrm{SIL}}$: cat. of first order fibrations with Cartesian closed base category
(morphism: pair of functors with suitable preservations)

Equivalence Theorem

The following is an (pseudo) adjunction whose counit is a natural equivalence; hence each $(p : \mathbb{E} \to \mathbb{B}) \in \mathrm{Fib}_{\mathrm{SIL}}$ is cannonically equivalent to $\mathcal{T}(\mathrm{Int}(p))$

$$\mathrm{Fib}_{\mathrm{SIL}} \underset{\mathcal{T}}{\overset{\mathrm{Int}}{\underset{\top}{\rightleftarrows}}} \mathrm{Spec}_{\mathrm{SIL}}$$

Example

Rep

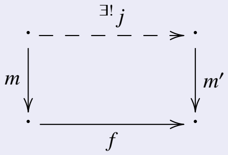
obj. : represented spaces

morph. : computable functions

Mono

obj. : monomorphisms in Rep

morph. : $m \xrightarrow{f} m'$ in Rep

$\iff$ 

in Rep

$\mathrm{cod}_{\mathrm{Rep}}$: Mono $\to$ Rep

obj. : $m \mapsto \mathrm{cod}\, m$

morph. : $f \mapsto f$

$\mathrm{cod}_{\mathrm{Rep}}$ is a first order fibration with Cartesian closed base category

Weihrauch Lattice: 1/4

Notation

(meta variable: $F, G, \cdots \subseteq \omega^\omega \times \omega^\omega$)

$$\begin{aligned} F[p] &:= \{q \in \omega^\omega : (p,q) \in F\} \\ \mathrm{supp}(F) &:= \{p \in \omega^\omega : F[p] \neq \emptyset\} \end{aligned}$$

Choice Function

A partial function f on Baire Space is a *choice function* of F iff $^\forall p \in \mathrm{supp}(F).\ f(p){\downarrow} \in F[p]$

Weihruach Reducibility

$F \leq_W G$ iff there is a pair k and l of computable partial functions on Baire space s.t. given a choice funciton g of G, $l\langle \mathrm{id}, gk\rangle$ is a choice function of F

$\mathfrak{W}_0$: the induced degree structure w.r.t. $\leq_W$

Weihrauch Lattice: 2/4

Double Negation Density

A formula $\varphi_{v_0:\sigma}$ in $\mathrm{Int}(\mathrm{cod}_{\mathsf{Rep}})$ is *double negation dense* iff $v_0:\sigma \mid \Lambda \vdash \neg\neg\varphi_{v_0:\sigma}$ is derivable over $\mathrm{Int}(\mathrm{cod}_{\mathsf{Rep}})$

Fibrewise Dual Redicibility

Let $\varphi_{v_0:\sigma}$ and $\psi_{v_0:\tau}$ be double negation dense over $\mathrm{Int}(\mathrm{cod}_{\mathsf{Rep}})$. Define:

$$\varphi_{v_0:\sigma} \leq^1 \psi_{v_0:\tau}$$

$\Longleftrightarrow$ there is a term $t^{\tau}_{v_0:\sigma}$ s.t. $v_0:\sigma \mid \psi_{v_0:\tau}[t^{\tau}_{v_0:\sigma}/v_0] \vdash \varphi_{v_0:\sigma}$ is derivable in $\mathrm{Int}(\mathrm{cod}_{\mathsf{Rep}})$

$\mathfrak{D}_{\top/\neg\neg}\mathsf{Rep}$: the induced degree structure w.r.t. $\leq^1$

Lemma 2 < Equivalence Theorem

$\mathfrak{W}_0$ has an embedding $\overline{\pi}$ into $\mathfrak{D}_{\top/\neg\neg}\mathtt{Rep}$ which has a right adjoint ϵ

$$\mathfrak{D}_{\top/\neg\neg}\mathtt{Rep} \; \underset{\overline{\pi}}{\overset{\epsilon}{\rightleftarrows}} \; \mathfrak{W}_0 \qquad (\epsilon \top \overline{\pi})$$

Define $\overline{\pi} : \mathfrak{W} \to \mathfrak{D}_{\top/\neg\neg}\mathtt{Rep}$ by $F \mapsto (v_0 : \overline{\mathscr{I}_F} \vdash \overline{\pi_F}(v_0) : \mathtt{Prop})$ where:

$$\begin{array}{rcll} \mathscr{I}_F & = & (\mathrm{supp}(F), \mathscr{I}_F) & \\ & & \mathscr{I}_F : p \mapsto p & \text{if } p \in \mathrm{supp}(F) \\ \mathscr{G}_F & = & (\mathrm{supp}(F), \mathscr{G}_F) & \\ & & \mathscr{I}_F : \langle p, q \rangle \mapsto p & \text{if } q \in F[p] \\ \pi_F & : & \mathscr{G}_F \to \mathscr{I}_F & \\ & & p \mapsto p & \end{array}$$

Lemma 3

The following square commute:

$$\begin{array}{ccc}
\mathfrak{D}_{\top/\neg\neg}\mathtt{Rep} & \xrightarrow{\ \overline{\omega}(-)\ } & \mathfrak{D}_{\top/\neg\neg}\mathtt{Rep} \\
\Big\downarrow{\scriptstyle \epsilon} & & \Big\downarrow{\scriptstyle \epsilon} \\
\mathfrak{W}_0 & \xrightarrow[\ !(-)\ ]{} & \mathfrak{W}_0
\end{array}$$

where $\overline{\omega}(-)$ is parametrization relative to (the name of) the natural number system and $!(-)$ is a closure operator on $\mathfrak{W}_0$, called *countable parallelization*, defined by:

$$!F : \langle p_0, p_1, \cdots \rangle \mapsto \{\langle q_0, q_1, \cdots \rangle : {}^{\forall} i \in \omega.\ q_i \in F[p_i]\}$$

Main Theorem

$\mathrm{HA}^{\lambda+}$: 1/2

Signature of $\mathrm{HA}^{\lambda+}$

Base Type Symbol: N for natural number system
2 for two elements boolean

Function Symbol : S for successor function
0_N for constants of type N
$0_2, 1_2$ for constants of type 2
E for embedding of 2 into N
R^σ for recursors

$\Delta_0(\Gamma)$-formula

$$\delta ::= \perp \mid t_\Gamma^N =_N t_\Gamma^N \mid t_\Gamma^2 =_2 t_\Gamma^2 \mid \delta \vee \delta \mid \delta \wedge \delta \mid \delta \to \delta \mid \exists n \leq t_\Gamma^N.\delta \mid \forall n \leq t_\Gamma^N.\delta$$

$\mathrm{HA}^{\lambda+}$: 2/2

Axioms of $\mathrm{HA}^{\lambda+}$

- Axioms for $S, 0_N$
- Axioms for E (and $0_2, 1_2$) as an embedding of 2 into N
- Axioms for R^σ as a recursor
- Induction Scheme:
  $$\Gamma \mid \varphi[0/n],\ \forall n{:}N.(\varphi \to \varphi[n+1/n]) \vdash \forall n{:}N.\varphi$$
  (where φ : ^{N}q.f.)
- Δ_0-Comprehension Scheme:
  $$\Gamma \mid \Lambda \vdash \exists p{:}(N \to 2).\forall n{:}N.(\delta \leftrightarrow pn =_2 1_2)$$
  (where δ : $\Delta_0(\Gamma, n{:}N)$-formula, p : the first flesh variable)
- Extensionality Scheme w.r.t. N:
  $$\Gamma \mid \forall n{:}N.(t_0 =_\sigma t_1) \vdash (\lambda n{:}N.t_0) =_{N\to\sigma} (\lambda n{:}N.t_1)$$

Main Theorem: 1/3

Main Theorem

There is a translation $^{\dagger}(-)$ from $\mathrm{HA}^{\lambda+}$ to $\mathrm{Int}(\mathrm{cod}_{\mathsf{Rep}})$
s.t. if $\Lambda \mid \forall y\!:\!\tau.\psi_0 \vdash \forall x\!:\!\sigma.\varphi_0$ is derivable over $\mathrm{HA}^{\lambda+}$,
then $![\![\varphi_0]\!]_{x:\sigma} \geq_W ![\![\varphi_0]\!]_{x:\sigma}$ in $\mathfrak{W}_0$

where:

- ψ_0 is ^{N}p.e.f. and ^{N}n.u.f.
- φ_0 is ^{N}n.e.f. and ^{N}p.u.f.
- $({}^{\dagger}\varphi_0)_{{}^{\dagger}x:{}^{\dagger}\sigma}$ and $({}^{\dagger}\psi_0)_{{}^{\dagger}y:{}^{\dagger}\tau}$ are double negation dense
- $[\![\varphi_0]\!]_{x:\sigma} \;:=\; \epsilon({}^{\dagger}\varphi_0)_{{}^{\dagger}x:{}^{\dagger}\sigma}$

Main Theorem: 2/3

Proof: 1/2

Define ${}^{\dagger}(-)$ by:

for base type : $N \mapsto \overline{\omega}$ where $\omega = (\omega, \delta_\omega),\ \delta_\omega : ip \mapsto i,$

$2 \mapsto \overline{2}$ where $2 = (2, \delta_2), \delta_2 : 0p \mapsto 0,\ 1p \mapsto 1$

for function symbol: $0_N \mapsto \overline{(0 : 1 \to \omega)}, S \mapsto \overline{(- + 1)}, 0_2 \mapsto \overline{(0 : 1 \to 2)},$

$1_2 \mapsto \overline{(1 : 1 \to 2)}, E \mapsto \overline{(\iota : 2 \to \omega)}, \cdots$ (Omit)

Define $\mathscr{A}$ as a subsystem of $\mathrm{Int}(\mathrm{cod}_{\mathsf{Rep}})$ by:

- Add tranlations (via ${}^{\dagger}(-)$) of axioms for $S, 0_N, E, 0_2, 1_2, R^\sigma,$ Induction Scheme and Extensionality Scheme
- Add ${}^{\dagger}\Gamma \mid \Lambda \vdash {}^{\dagger}(\forall n{:}N.(\delta \leftrightarrow pn =_2 1_2))[t_{{}^{\dagger}\Gamma}^{{}^{\dagger}(N \to 2)} / {}^{\dagger}p]$ iff it is an axiom of $\mathrm{Int}(\mathrm{cod}_{\mathsf{Rep}})$ and δ is $\Delta_0(\Gamma, n{:}N)$-formula

Note that:

- $\mathscr{A}$ is ${}^{\overline{\omega}}$p.e.f. and ${}^{\overline{\omega}}$n.u.f.
- if $\Gamma \mid \Theta \vdash \varphi$ is deriv. over $\mathrm{HA}^{\lambda+}$, then ${}^{\dagger}\Gamma \mid {}^{\dagger}\Theta \vdash {}^{\dagger}\varphi$ is deriv. over $\mathscr{A}$

Main Theorem: 3/3

Proof: 2/2

We obtain:

$$\begin{array}{lll}
& \Lambda \mid \forall y{:}\tau.\psi_0 \vdash \forall x{:}\sigma.\varphi_0 & \text{is derivable over } \mathrm{HA}^{\lambda+} \\
\Longleftrightarrow & \Lambda \mid \forall y{:}\tau.\psi_0 \vdash {}^N\forall x{:}\sigma.\varphi_0 & \text{is derivable over } \mathrm{HA}^{\lambda+} \\
\Longrightarrow & \Lambda \mid {}^\dagger\forall y{:}\tau.\psi_0 \vdash {}^{\overline{\omega}\dagger}\forall x{:}\sigma.\varphi_0 & \text{is derivable over } \mathscr{A} \\
\Longleftrightarrow & {}^\dagger x{:}\overline{\omega}\to{}^\dagger\sigma \mid {}^\dagger\forall y{:}\tau.\psi_0 \vdash \forall z{:}\overline{\omega}.{}^\dagger\varphi_0[{}^\dagger xz/{}^\dagger x] & \text{is derivable over } \mathscr{A} \\
\Longleftrightarrow & {}^\dagger x{:}\overline{\omega}\to{}^\dagger\sigma \mid (\forall w{:}\overline{\omega}.{}^\dagger\psi_0[{}^\dagger yw/{}^\dagger y])[t^{\overline{\omega}\to{}^\dagger\tau}_{{}^\dagger x{:}{}^\dagger\sigma}/{}^\dagger y] \vdash \forall z{:}\overline{\omega}.{}^\dagger\varphi_0[{}^\dagger xz/{}^\dagger x] & \\
\end{array}$$

is derivable over $\mathscr{A}$ for some $t^{\overline{\omega}\to{}^\dagger\tau}_{{}^\dagger x{:}{}^\dagger\sigma}$ (Lemma 1 $<$ Cut-Elim. Thm + α)

$$\begin{array}{lll}
\Longrightarrow & {}^{\overline{\omega}}(({}^\dagger\psi_0)_{{}^\dagger y{:}{}^\dagger\tau})^1 \geq {}^{\overline{\omega}}(({}^\dagger\varphi_0)_{{}^\dagger x{:}{}^\dagger\sigma}) & \\
\Longrightarrow & !\epsilon({}^\dagger\psi_0)_{{}^\dagger y{:}{}^\dagger\tau} \geq_W !\epsilon({}^\dagger\varphi_0)_{{}^\dagger x{:}{}^\dagger\sigma} & \text{(Lemma 2 \& 3 } < \text{ Equiv. Thm)} \\
\Longleftrightarrow & ![\![\varphi_0]\!]_{x:\sigma} \geq_W ![\![\varphi_0]\!]_{x:\sigma} &
\end{array}$$

Application: 1/2

LPO

$$\mathrm{LPO} \;:\equiv\; \Gamma \mid \Lambda \vdash \exists n{:}N.\delta \vee \neg\exists n{:}N.\delta \qquad (\delta :\; \Delta_0(\Gamma, n{:}N)\text{-formula})$$

$$\mathrm{LPO}_0 \;:\equiv\; \exists n{:}N.pn =_2 1_2 \vee \neg\exists n{:}N.pn =_2 1_2$$

$$\mathtt{LPO} \;:\; \begin{array}{l} p \mapsto \{0q : q \in 2^\omega\} \text{ if } p \neq 0^\omega \\ 0^\omega \mapsto \{1q : q \in 2^\omega\} \end{array}$$

LLPO

$$\mathrm{LLPO} \;:\equiv\; \Gamma \mid \neg(\exists n{:}N.\delta_0 \wedge \exists n{:}N.\delta_1) \vdash \neg\exists n{:}N.\delta_0 \vee \neg\exists n{:}N.\delta_1$$

$$(\delta_0, \delta_1 :\; \Delta_0(\Gamma, n{:}N)\text{-formula})$$

$$\mathrm{LLPO}_0 \;:\equiv\; \neg(\exists n{:}N.p(2n) =_2 1_2 \wedge \exists n{:}N.p(2n+1) =_2 1_2)$$

$$\rightarrow \neg\exists n{:}N.p(2n) =_2 1_2 \vee \neg\exists n{:}N.p(2n+1) =_2 1_2$$

$$\mathtt{LLPO} \;:\; p = 0^*(0+1)0^\omega \mapsto \quad \{0q : q \in 2^\omega,\; p(2- \quad) = 0^\omega\}$$

$$\cup\{1q : q \in 2^\omega,\; p(2 - +1) = 0^\omega\}$$

Application: 2/2

Proposition

Over $HA^{\lambda+}$, LLPO does not imply LPO

We obtain:

$$
\begin{array}{ll}
 & \text{LLPO implies LPO over } HA^{\lambda+} \\
\iff & \Lambda \mid \forall p:(N\to 2).\mathrm{LLPO}_0 \vdash \forall p:(N\to 2).\mathrm{LPO}_0 \text{ is derivable over } HA^{\lambda+} \\
\implies & ![\![\mathrm{LLPO}_0]\!]_{p:N\to 2} \geq_W ![\![\mathrm{LPO}_0]\!]_{p:N\to 2} \qquad \text{(Main Theorem)} \\
\iff & !\mathtt{LLPO} \geq_W !\mathtt{LPO}
\end{array}
$$

However $!\mathtt{LLPO} \not\geq_W !\mathtt{LPO}$ (V. Brattka & G. Gherardi, 2011).

Thank you for listening