From Well-Quasi-Orders to Noetherian Spaces: the Reverse Mathematics Viewpoint

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Well quasi-orders

1. Well quasi-orders

2. From well quasi-orders to Noetherian spaces

3. Coding and the forward directions
   Working with $\mathcal{U}(\mathcal{P}_f^b(Q))$ and $\mathcal{U}(\mathcal{P}_f^\#(Q))$
   Working with $\mathcal{U}(\mathcal{P}_f^b(Q))$ and $\mathcal{U}(\mathcal{P}_f^\#(Q))$

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Well quasi-orders

A quasi-order is a binary relation which is reflexive and transitive (no antisymmetry).

A quasi-order \( Q = (Q, \leq_Q) \) is a well quasi-order (wqo) if for every \( f : \mathbb{N} \to Q \) there exists \( i < j \) such that \( f(i) \leq_Q f(j) \).

There are many equivalent characterizations of wqos:

- \( Q \) is well-founded and has no infinite antichains;
- every sequence in \( Q \) has a weakly increasing subsequence;
- every nonempty subset of \( Q \) has a finite set of minimal elements;
- all linear extensions of \( Q \) are well orders.

The reverse mathematics and computability theory of these equivalences has been studied in (Cholak-M-Solomon 2004).

All equivalences are provable in WKL\(_0+\)CAC.
Some examples of wqos

- Finite partial orders
- Well-orders
- Finite strings over a finite alphabet (Higman, 1952)
- Finite trees (Kruskal, 1960)
- Transfinite sequences with finite labels (Nash-Williams, 1965)
- Countable linear orders (Laver 1971, proving Fraïssé’s conjecture)
- Finite graphs (Robertson and Seymour, 2004)

The ordering is some kind of embeddability
Closure properties of wqos

- The sum and disjoint sum of two wqos are wqos
- The product of two wqos is a wqo
- Finite strings over a wqo are a wqo (Higman, 1952)
- Finite trees with labels from a wqo are a wqo (Kruskal, 1960)
- Transfinite sequences with labels from a wqo which use only finitely many labels are a wqo (Nash-Williams, 1965)
Quasi-orders on the powerset

Let \( Q = (Q, \leq_Q) \) be a quasi-order. For \( A, B \in \mathcal{P}(Q) \):

\[
\begin{align*}
A \leq^b B & \iff \forall a \in A \exists b \in B a \leq_Q b \iff A \subseteq B \downarrow \\
A \leq^\# B & \iff \forall b \in B \exists a \in A a \leq_Q b \iff B \subseteq A \uparrow
\end{align*}
\]

Let \( \mathcal{P}^b(Q) = (\mathcal{P}(Q), \leq^b) \) and \( \mathcal{P}^\#(Q) = (\mathcal{P}(Q), \leq^\#) \).

\( \mathcal{P}^b_f(Q) \) and \( \mathcal{P}^\#_f(Q) \) are the restrictions to finite subsets of \( Q \).

**Theorem (Erdős–Rado 1952)**

\( Q \) is wqo if and only if \( \mathcal{P}^b_f(Q) \) is wqo.

\( Q \) wqo does not imply that any of \( \mathcal{P}^b(Q), \mathcal{P}^\#(Q) \) and \( \mathcal{P}^\#_f(Q) \) are wqo.
The reverse mathematics of the Erdős–Rado theorem

**Theorem (RCA$_0$)**

The following are equivalent:

(i) ACA$_0$;

(ii) if $Q$ is wqo, then $P^{b}_f(Q)$ is wqo.
From well quasi-orders to Noetherian spaces

1. Well quasi-orders

2. From well quasi-orders to Noetherian spaces

3. Coding and the forward directions
   - Working with $\mathcal{U}(P_f^b(Q))$ and $\mathcal{U}(P_f^\#(Q))$
   - Working with $\mathcal{U}(P_f^b(Q))$ and $\mathcal{U}(P_f^\#(Q))$

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6. Finer analysis?
A topological space $X$ is **Noetherian** if every open subset of $X$ is compact.

Some equivalent characterizations of Noetherian spaces:

- every subset of $X$ is compact;
- every increasing sequence of open subsets of $X$ stabilizes;
- every decreasing sequence of closed subsets of $X$ stabilizes.

Noetherian spaces are important in algebraic geometry: the set of prime ideals (aka the spectrum) of a Noetherian ring with the Zariski topology is a Noetherian space.

If a $T_2$ space is Noetherian then it is finite.
Let $Q = (Q, \leq_Q)$ be a quasi-order.

The **Alexandroff topology** $\mathcal{A}(Q)$ is the topology on $Q$ with the downward closed subsets of $Q$ as closed sets.

The **upper topology** $\mathcal{U}(Q)$ is the topology on $Q$ with the downward closures of finite subsets of $Q$ as a basis for the closed sets.

**Why these two topologies?**

Given a topological space, define a quasi-order on the points by

$$x \preceq y \iff \text{every open set that contains } x \text{ also contains } y.$$ 

$\mathcal{A}(Q)$ is the finest topology on $Q$ such that $\preceq$ is $\leq_Q$.

$\mathcal{U}(Q)$ is the coarsest such topology.

If $Q$ is not an antichain $\mathcal{A}(Q)$ and $\mathcal{U}(Q)$ are not $T_1$. 

Which features of the quasi-order $Q$ are reflected in $A(Q)$ and $U(Q)$?

**Fact**

$Q$ is wqo if and only if $A(Q)$ is Noetherian.

If $Q$ is wqo then $U(Q)$ is Noetherian.

Recall that by Erdős and Rado if $Q$ is wqo, then $P_f^b(Q)$ is a wqo. Thus if $Q$ is wqo, then $U(P_f^b(Q))$ is Noetherian.

However $U(Q)$ might be Noetherian even when $Q$ is not wqo.
$U(Q)$ might be Noetherian even when $Q$ is not wqo.

If $Q$ is wqo then $P^b(Q)$, $P^f_#(Q)$ and $P^h_#(Q)$ are not necessarily wqo.

**Theorem (Goubault-Larrecq, 2007)**

*If $Q$ is wqo then $U(P^b(Q))$ and $U(P^f_#(Q))$ are Noetherian.*

If $Q$ is wqo, for every $A \in P(Q)$ there is a $B \in P_f(Q)$ such that $A \equiv_# B$. Thus the theorem implies that if $Q$ is wqo, then $U(P^h_#(Q))$ is Noetherian.

In a subsequent paper Goubault-Larrecq applied his theorem to infinite-state verification problems.

We want to study the reverse mathematics of Goubault-Larrecq’s theorem.
Coding and the forward directions

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What topological spaces do we need to code?

1. $U(P^b(Q))$
2. $U(P^b_f(Q))$
3. $U(P^\#(Q))$
4. $U(P^\#_f(Q))$

Assuming that $Q$ is countably infinite

$U(P^b_f(Q))$ and $U(P^\#_f(Q))$ are countable spaces with a countable basis;

$U(P^b(Q))$ and $U(P^\#(Q))$ are uncountable spaces and we described their
topology using an uncountable basis.
Countable second countable spaces

Dorais introduced a framework for dealing with countable second countable spaces.

**Definition (RCA₀)**

A *countable second-countable space* consists of a set $X$, a sequence $(U_i)_{i \in I}$ of subsets of $X$, and a function $k : X \times I \times I \to I$ such that

- if $x \in X$, then $x \in U_i$ for some $i \in I$;
- if $x \in U_i \cap U_j$, then $x \in U_k(x,i,j) \subseteq U_i \cap U_j$. 
Coding open sets and expressing compactness

Every function \( h : \mathbb{N} \rightarrow \mathcal{P}_f(I) \) codes the open set \( G_h = \bigcup_{n \in \mathbb{N}} \bigcup_{i \in h(n)} U_i \).

**Definition (RCA\(_0\))**

The open set \( G_h \) is **compact** if for every \( f : \mathbb{N} \rightarrow \mathcal{P}_f(I) \) with \( G_h \subseteq \bigcup_{n \in \mathbb{N}} \bigcup_{i \in f(n)} U_i \), there exists \( N \) such that \( G_h \subseteq \bigcup_{n < N} \bigcup_{i \in f(n)} U_i \).
Equivalent definitions of Noetherian are equivalent

**Lemma (RCA₀)**

For a countable second-countable space \((X, (U_i)_{i \in I}, k)\), the following statements are equivalent:

(i) every open set is compact;

(ii) for every open set \(G_h\), there exists \(N\) such that \(G_h = \bigcup_{n < N} \bigcup_{i \in h(n)} U_i\);

(iii) for every sequence \((G_n)_{n \in \mathbb{N}}\) of open sets such that \(\forall n G_n \subseteq G_{n+1}\), there exists \(N\) such that \(\forall n > N G_n = G_N\);

(iv) for every sequence \((F_n)_{n \in \mathbb{N}}\) of closed sets such that \(\forall n F_n \supseteq F_{n+1}\), there exists \(N\) such that \(\forall n > N F_n = F_N\).

**Definition (RCA₀)**

A countable second-countable space is *Noetherian* if it satisfies any of the equivalent conditions above.
Coding the Alexandroff and upper topologies

**Definition (RCA₀)**

Let \( Q \) be a quasi-order.

- A base for the Alexandroff topology on \( Q \) is given by \((U_q)_{q \in Q}\), where \( U_q = q^{\uparrow} \) for each \( q \in Q \), and \( k(q, p, r) = q \).
  Let \( A(Q) \) denote the countable second-countable space \((Q, (U_q)_{q \in Q}, k)\).

- A base for the upper topology on \( Q \) is given by \((V_i)_{i \in \mathcal{P}_f(Q)}\), where \( V_i = Q \setminus (i^{\downarrow}) \) for each \( i \in \mathcal{P}_f(Q) \), and \( \ell(q, i, j) = i \cup j \).
  Let \( U(Q) \) denote the countable second-countable space \((Q, (V_i)_{i \in \mathcal{P}_f(Q)}, \ell)\).
Basic facts

**Lemma (RCA₀)**

Let $Q$ be a quasi-order.

(i) If $\mathcal{A}(Q)$ Noetherian, then $U(Q)$ Noetherian.

(ii) $Q$ is wqo if and only if $\mathcal{A}(Q)$ is Noetherian.

**Corollary (ACA₀)**

If $Q$ is wqo then $\mathcal{A}(P^b_f(Q))$ and $U(P^b_f(Q))$ are Noetherian.

We can also express “if $Q$ is wqo then $U(P^#_f(Q))$ is Noetherian” in RCA₀.
\( \mathcal{U}(\mathcal{P}^b(Q)) \) and \( \mathcal{U}(\mathcal{P}^\#(Q)) \) are second countable

\( \mathcal{U}(\mathcal{P}^b(Q)) \) and \( \mathcal{U}(\mathcal{P}^\#(Q)) \) are spaces with uncountably many points. Moreover we described their topology using uncountable basis. However both spaces have (non-obvious) countable basis.

**Fact**

The sets of the form \( \{ Q \setminus (q \uparrow) \mid q \in i \} \downarrow^b \), where \( i \in \mathcal{P}_f(Q) \), are a basis for the closed sets of the topology of \( \mathcal{U}(\mathcal{P}^b(Q)) \).

**Fact**

The sets of the form \( \{ \{q\} \mid q \in i \} \downarrow^\# \), where \( i \in \mathcal{P}_f(Q) \), are a basis for the closed sets of the topology of \( \mathcal{U}(\mathcal{P}^\#(Q)) \).
Where is second countability provable?

**Lemma (RCA₀)**

The following are equivalent:

(i) ACA₀;

(ii) If Q is a quasi-order and E ⊆ Q, then \( \{E\} \downarrow \) is a countable intersection of sets of the form \( \{Q \setminus (q \uparrow) \mid q \in i\} \downarrow \), with i ∈ \( \mathcal{P}_f(Q) \);

(iii) the same statement when Q is a well order.

**Lemma (RCA₀)**

If Q is a quasi-order and E ⊆ Q, then \( \{E\} \downarrow \) is a countable intersection of sets of the form \( \{\{q\} \mid q \in i\} \downarrow \), with i ∈ \( \mathcal{P}_f(Q) \).
A scheme for representing uncountable second-countable spaces

A *second-countable space* is coded by a set $I \subseteq \mathbb{N}$ and formulas $\varphi(X)$, $\Psi_\equiv(X, Y)$, and $\Psi_\in(X, n)$

$I$ is the set of codes for open sets

$\varphi(X)$ means “$X$ codes a point”

$\Psi_\equiv(X, Y)$ means “$X$ and $Y$ code the same point”

$\Psi_\in(X, i)$ means “the point coded by $X$ belongs to the open set coded by $i \in I$”

We ask that

- if $\varphi(X)$, then $\Psi_\in(X, i)$ for some $i \in I$;
- if $\varphi(X)$, $\Psi_\in(X, i)$, and $\Psi_\in(X, j)$ for some $i, j \in I$, then there exists $k \in I$ such that $\Psi_\in(X, k)$ and

  $\forall Y[\Psi_\in(Y, k) \implies (\Psi_\in(Y, i) \land \Psi_\in(Y, j))]$;
- if $\varphi(X)$, $\varphi(Y)$, $\Psi_\in(X, i)$ for some $i \in I$, and $\Psi_\equiv(X, Y)$, then $\Psi_\in(Y, i)$.
Old codings of spaces fit in this scheme

When we code a complete separable metric space \((A, d)\) using a countable dense set \(A\), we let \(I = A \times \mathbb{Q}^+\) and then set

\[\varphi(X) \overset{\text{def}}{=} \text{“}X\text{ is a rapidly converging Cauchy sequence of points in } A\text{”}\]

\[\Psi_{=}(X, Y) \overset{\text{def}}{=} \text{“}the distances between the points of the sequences } X \text{ and } Y \text{ go to } 0 \text{ fast enough}\]

\[\Psi_{\in}(X, (a, q)) \overset{\text{def}}{=} \text{“}the distance between the point coded by } X \text{ and } a \in A \text{ is less than } q \in \mathbb{Q}^+\text{”}\]

Also, Mummert’s MF spaces (second-countable \(T_1\) spaces with the strong Choquet property) can be accommodated by our scheme.
The codings for $U(\mathcal{P}^b(Q))$ and $U(\mathcal{P}^\#(Q))$

**Definition (RCA$_0$)**

Let $Q$ be a quasi-order. The second-countable space $U(\mathcal{P}^b(Q))$ is coded by the set $I = \mathcal{P}_f(Q)$ and the formulas:

- $\varphi(X) \overset{\text{def}}{=} X \subseteq Q$;
- $\Psi_\preceq(X, Y) \overset{\text{def}}{=} X = Y$;
- $\Psi_\in(X, i) \overset{\text{def}}{=} i \subseteq X\downarrow$.

The second-countable space $U(\mathcal{P}^\#(Q))$ is coded by the set $I = \mathcal{P}_f(Q)$ and the formulas:

- $\varphi(X) \overset{\text{def}}{=} X \subseteq Q$;
- $\Psi_\preceq(X, Y) \overset{\text{def}}{=} X = Y$;
- $\Psi_\in(X, i) \overset{\text{def}}{=} i \cap X\uparrow = \emptyset$. 
Relations between topologies

Using these codings we can formalize the statements “\( U(P^b(Q)) \) is Noetherian” and “\( U(P^\#(Q)) \) is Noetherian” (the equivalence of the various definitions is provable in \( \text{RCA}_0 \)).

In general, \( U(P_f^b(Q)) \) is strictly coarser than the subspace topology on \( P_f(Q) \) induced by \( U(P^b(Q)) \).

However, \( U(P_f^\#(Q)) \) is the subspace topology on \( P_f(Q) \) induced by \( U(P^\#(Q)) \).

**Theorem (RCA\(_0\))**

*Let \( Q \) be a quasi-order.*

1. If \( U(P^b(Q)) \) is Noetherian, then \( U(P_f^b(Q)) \) is Noetherian.
2. If \( U(P^\#(Q)) \) is Noetherian, then \( U(P_f^\#(Q)) \) is Noetherian.

This is not entirely trivial because the codings are different!
Proving Goubault-Larrecq’s theorems

**Theorem (ACA₀)**

If \( Q \) is wqo then \( \mathcal{U}(\mathcal{P}^b(\mathcal{Q})) \) and \( \mathcal{U}(\mathcal{P}^\#(\mathcal{Q})) \) are Noetherian.

Goubault-Larrecq’s original proofs are category-theoretic. We need to use completely different, more elementary, arguments.

**Corollary (ACA₀)**

If \( Q \) is wqo then \( \mathcal{U}(\mathcal{P}^b_f(\mathcal{Q})) \) and \( \mathcal{U}(\mathcal{P}^\#_f(\mathcal{Q})) \) are Noetherian.
The reversals

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6. Finer analysis?
We want to show that

if $Q$ is a wqo, then $U(P_f^*(Q))$ is Noetherian

implies ACA$_0$ over RCA$_0$ (where $\star \in \{\flat, \sharp\}$).

Our strategy is to produce, given an injective $f : \mathbb{N} \to \mathbb{N}$, a $f$-computable $Q$ such that RCA$_0$ proves:

- $U(P_f^*(Q))$ is not Noetherian;
- if $g$ is a bad sequence in $Q$, then $g \oplus f$ computes $\text{ran}(f)$.
The reversals

True and false stages of $f$

$f : \mathbb{N} \to \mathbb{N}$ is injective

- $n$ is $f$-true if $\forall k > n \ f(n) < f(k)$;
- $n$ is $f$-true at stage $s$ if $n < s$ and $\forall k \ (n < k \leq s \implies f(n) < f(k))$.

Otherwise $n$ is false (at stage $s$).

If $g$ is an injective sequence of true numbers, then $\text{ran}(f) \leq_T g \oplus f$ because we may assume that $g$ is strictly increasing and then

$$k \in \text{ran}(f) \iff \exists n \leq g(k) \ f(n) = k.$$
A pseudo well order

\( f : \mathbb{N} \to \mathbb{N} \) is injective

The prototype of a construction using true and false stages produces a linear order \( \mathcal{L} \) such that

- \( \mathcal{L} \) has order type \( \omega + \omega^* \);
- the \( \omega \) part of \( \mathcal{L} \) consists of the \( f \)-false stages and is \( \Sigma^0_1 \) in \( f \);
- the \( \omega^* \) part of \( \mathcal{L} \) consists of the \( f \)-true stages and is \( \Pi^0_1 \) in \( f \).

Thus, if we know that \( \mathcal{L} \) is not a well order then we can compute \( \text{ran}(f) \).

\( \mathcal{L} \) is defined recursively: we put the new element \( s \) below the \( n \)'s that are \( f \)-true at stage \( s \) and above the \( n \)'s that are \( f \)-false at stage \( s \).
Generalizing the construction

\[ f : \mathbb{N} \to \mathbb{N} \text{ is injective} \]

We generalize the previous construction: rather then adding one element, at each stage we add a finite partial order \( \mathcal{R} \) with a designated point \( x \).

By controlling how the \( s \)-th copy of \( \mathcal{R} \) sits into the construction (depending on the \( n \)'s that are \( f \)-true and \( f \)-false at stage \( s \)) we define a partial order \( \Xi_f(\mathcal{R}, x) \) so that

**Lemma \( \text{(RCA}_0 \text{)} \)**

*If \( \Xi_f(\mathcal{R}, x) \) is not a wqo then \( \text{ran}(f) \) exists.*
The reversals

\( f : \mathbb{N} \to \mathbb{N} \) is injective

Making appropriate choices of \( R \) and \( x \) we build \( Q = \Xi_f(R, x) \) such that \( \mathcal{U}(\mathcal{P}_f^b(Q)) \) is not Noetherian and obtain

**Theorem (RCA\(_0\))**

The statement “if \( Q \) is wqo then \( \mathcal{U}(\mathcal{P}_f^b(Q)) \) is Noetherian” implies ACA\(_0\).

Using a different \( R \) we get

**Theorem (RCA\(_0\))**

The statement “if \( Q \) is wqo then \( \mathcal{U}(\mathcal{P}_f^h(Q)) \) is Noetherian” implies ACA\(_0\).
The main result

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Summing up: the reverse mathematics of Goubault-Larrecq’s theorems

Main Theorem ($\text{RCA}_0$)

The following are equivalent:

(i) $\text{ACA}_0$;

(ii) if $Q$ is wqo then $\mathcal{A}(\mathcal{P}_f^b(Q))$ is Noetherian;

(iii) if $Q$ is wqo then $\mathcal{U}(\mathcal{P}_f^b(Q))$ is Noetherian;

(iv) if $Q$ is wqo then $\mathcal{U}(\mathcal{P}_f^#(Q))$ is Noetherian;

(v) if $Q$ is wqo then $\mathcal{U}(\mathcal{P}_f^b(Q))$ is Noetherian;

(vi) if $Q$ is wqo then $\mathcal{U}(\mathcal{P}_f^#(Q))$ is Noetherian.
Finer analysis?

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6. Finer analysis?
Many theorems studied in reverse mathematics are $\Pi^1_2$ statements of the form

$$\forall X (\Phi(X) \implies \exists Y \Psi(X,Y))$$

where $\Phi$ and $\Psi$ are arithmetical.

In this situation we often say that an $X$ such that $\Phi(X)$ is a problem, and a $Y$ satisfying $\Psi(X,Y)$ is a solution to the problem.

We look at the multi-valued map assigning to a problem the set of its solutions.

We compare these multi-valued maps using (strong) Weihrauch reducibility and/or (strong) reducibility.

These reductions lead to a finer analysis of the strength of the statements.
Goubault-Larrecq’s theorems are indeed $\Pi^1_2$ statements, but they are of the following form:

$$\forall X (\forall Z \Phi(X, Z) \implies \forall Y \Psi(X, Y))$$

with $\Phi$ and $\Psi$ arithmetical.

In fact both “$Q$ is wqo” and “$U(Q)$ is Noetherian” are $\Pi^1_1$.

These statements do not fit nicely into the problem/solution pattern.

We can rewrite them as

$$\forall X \forall Y (\neg \Psi(X, Y) \implies \exists Z \neg \Phi(X, Z)).$$

A problem is a pair consisting of a quasi-order $Q$ and a witness to the fact that $U(\mathcal{P}_f^b(Q))$ is not Noetherian.

Its solutions are the bad sequences in $Q$. 
Which is the real form of Goubault-Larrecq’s theorems?

In fact our proofs of both directions of the reverse mathematics results actually consider statements such as

\[ \text{if } \mathcal{U}(\mathcal{P}_{f}^{b}(Q)) \text{ is not Noetherian then } Q \text{ is not wqo} \]
Thank you for your attention!