From Well-Quasi-Orders to Noetherian Spaces: the Reverse Mathematics Viewpoint

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Outline

1. Well quasi-orders
2. From well quasi-orders to Noetherian spaces
3. Coding and the forward directions
4. The reversals
5. The main result
6. Finer analysis?

Well quasi-orders

1. **Well quasi-orders**
2. From well quasi-orders to Noetherian spaces
3. Coding and the forward directions
   - Working with $\mathcal{U}(\mathcal{P}^\flat_f(\mathcal{Q}))$ and $\mathcal{U}(\mathcal{P}^\sharp_f(\mathcal{Q}))$
   - Working with $\mathcal{U}(\mathcal{P}^\flat(\mathcal{Q}))$ and $\mathcal{U}(\mathcal{P}^\sharp(\mathcal{Q}))$
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Well quasi-orders

A quasi-order is a binary relation which is reflexive and transitive (no antisymmetry).

A quasi-order $\mathcal{Q} = (Q, \leq_Q)$ is a well quasi-order (wqo) if for every $f : \mathbb{N} \to Q$ there exists $i < j$ such that $f(i) \leq_Q f(j)$.

There are many equivalent characterizations of wqos:

- $\mathcal{Q}$ is well-founded and has no infinite antichains;
- every sequence in Q has a weakly increasing subsequence;
- every nonempty subset of Q has a finite set of minimal elements;
- all linear extensions of $\mathcal{Q}$ are well orders.

The reverse mathematics and computability theory of these equivalences has been studied in (Cholak-M-Solomon 2004).

All equivalences are provable in $\mathsf{WKL}_0 + \mathsf{CAC}$.

Some examples of wqos

- Finite partial orders
- Well-orders
- Finite strings over a finite alphabet (Higman, 1952)
- Finite trees (Kruskal, 1960)
- Transfinite sequences with finite labels (Nash-Williams, 1965)
- Countable linear orders (Laver 1971, proving Fraïssé's conjecture)
- Finite graphs (Robertson and Seymour, 2004)

The ordering is some kind of embeddability

Closure properties of wqos

- The sum and disjoint sum of two wqos are wqos
- The product of two wqos is a wqo
- Finite strings over a wqo are a wqo (Higman, 1952)
- Finite trees with labels from a wqo are a wqo (Kruskal, 1960)
- Transfinite sequences with labels from a wqo which use only finitely many labels are a wqo (Nash-Williams, 1965)

Quasi-orders on the powerset

Let $\mathcal{Q} = (Q, \leq_Q)$ be a quasi-order. For $A, B \in \mathcal{P}(Q)$:

$$A \leq^\flat B \iff \forall a \in A\, \exists b \in B\; a \leq_Q b \iff A \subseteq B{\downarrow}$$
$$A \leq^\sharp B \iff \forall b \in B\, \exists a \in A\; a \leq_Q b \iff B \subseteq A{\uparrow}$$

Let $\mathcal{P}^\flat(\mathcal{Q}) = (\mathcal{P}(Q), \leq^\flat)$ and $\mathcal{P}^\sharp(\mathcal{Q}) = (\mathcal{P}(Q), \leq^\sharp)$.
$\mathcal{P}^\flat_{\mathrm{f}}(\mathcal{Q})$ and $\mathcal{P}^\sharp_{\mathrm{f}}(\mathcal{Q})$ are the restrictions to finite subsets of Q.

Theorem (Erdős–Rado 1952)

$\mathcal{Q}$ is wqo if and only if $\mathcal{P}^\flat_{\mathrm{f}}(\mathcal{Q})$ is wqo.

$\mathcal{Q}$ wqo does not imply that any of $\mathcal{P}^\flat(\mathcal{Q})$, $\mathcal{P}^\sharp(\mathcal{Q})$ and $\mathcal{P}^\sharp_{\mathrm{f}}(\mathcal{Q})$ are wqo.

The reverse mathematics of the Erdős–Rado theorem

Theorem (RCA_0)

The following are equivalent:

(i) ACA_0;

(ii) *if* $\mathcal{Q}$ *is wqo, then* $\mathcal{P}_{\mathrm{f}}^{\flat}(\mathcal{Q})$ *is wqo.*

From well quasi-orders to Noetherian spaces

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2. **From well quasi-orders to Noetherian spaces**
3. Coding and the forward directions
   - Working with $\mathcal{U}(\mathcal{P}^{\flat}_{\mathrm{f}}(\mathcal{Q}))$ and $\mathcal{U}(\mathcal{P}^{\sharp}_{\mathrm{f}}(\mathcal{Q}))$
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Noetherian spaces

A topological space X is Noetherian if every open subset of X is compact.

Some equivalent characterizations of Noetherian spaces:

- every subset of X is compact;
- every increasing sequence of open subsets of X stabilizes;
- every decreasing sequence of closed subsets of X stabilizes.

Noetherian spaces are important in algebraic geometry:
the set of prime ideals (aka the spectrum) of a Noetherian ring with the Zariski topology is a Noetherian space.

If a T_2 space is Noetherian then it is finite.

From quasi-orders to topological spaces

Let $\mathcal{Q} = (Q, \leq_Q)$ be a quasi-order.

The Alexandroff topology $\mathcal{A}(\mathcal{Q})$ is the topology on Q with the downward closed subsets of Q as closed sets.

The upper topology $\mathcal{U}(\mathcal{Q})$ is the topology on Q with the downward closures of finite subsets of Q as a basis for the closed sets.

Why these two topologies?
Given a topological space, define a quasi-order on the points by

$$x \preceq y \iff \text{every open set that contains } x \text{ also contains } y.$$

$\mathcal{A}(\mathcal{Q})$ is the finest topology on Q such that $\preceq$ is $\leq_Q$.
$\mathcal{U}(\mathcal{Q})$ is the coarsest such topology.

If $\mathcal{Q}$ is not an antichain $\mathcal{A}(\mathcal{Q})$ and $\mathcal{U}(\mathcal{Q})$ are not T_1.

Which features of the quasi-order $\mathcal{Q}$ are reflected in $\mathcal{A}(\mathcal{Q})$ and $\mathcal{U}(\mathcal{Q})$?

Fact

$\mathcal{Q}$ is wqo if and only if $\mathcal{A}(\mathcal{Q})$ is Noetherian.
If $\mathcal{Q}$ is wqo then $\mathcal{U}(\mathcal{Q})$ is Noetherian.

Recall that by Erdős and Rado if $\mathcal{Q}$ is wqo, then $\mathcal{P}_{\mathrm{f}}^{\flat}(\mathcal{Q})$ is a wqo.
Thus if $\mathcal{Q}$ is wqo, then $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^{\flat}(\mathcal{Q}))$ is Noetherian.

However $\mathcal{U}(\mathcal{Q})$ might be Noetherian even when $\mathcal{Q}$ is not wqo.

From well quasi-orders to Noetherian spaces

$\mathcal{U}(\mathcal{Q})$ might be Noetherian even when $\mathcal{Q}$ is not wqo.

If $\mathcal{Q}$ is wqo then $\mathcal{P}^\flat(\mathcal{Q})$, $\mathcal{P}^\sharp_\mathrm{f}(\mathcal{Q})$ and $\mathcal{P}^\sharp(\mathcal{Q})$ are not necessarily wqo.

Theorem (Goubault-Larrecq, 2007)

If $\mathcal{Q}$ is wqo then $\mathcal{U}(\mathcal{P}^\flat(\mathcal{Q}))$ and $\mathcal{U}(\mathcal{P}^\sharp_\mathrm{f}(\mathcal{Q}))$ are Noetherian.

If $\mathcal{Q}$ is wqo, for every $A \in \mathcal{P}(Q)$ there is a $B \in \mathcal{P}_\mathrm{f}(Q)$ such that $A \equiv^\sharp B$. Thus the theorem implies that if $\mathcal{Q}$ is wqo, then $\mathcal{U}(\mathcal{P}^\sharp(\mathcal{Q}))$ is Noetherian.

In a subsequent paper Goubault-Larrecq applied his theorem to infinite-state verification problems.

We want to study the reverse mathematics of Goubault-Larrecq's theorem.

Coding and the forward directions

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What topological spaces do we need to code?

1. $\mathcal{U}(\mathcal{P}^\flat(\mathcal{Q}))$
2. $\mathcal{U}(\mathcal{P}^\flat_{\mathrm{f}}(\mathcal{Q}))$
3. $\mathcal{U}(\mathcal{P}^\sharp(\mathcal{Q}))$
4. $\mathcal{U}(\mathcal{P}^\sharp_{\mathrm{f}}(\mathcal{Q}))$

Assuming that Q is countably infinite

$\mathcal{U}(\mathcal{P}^\flat_{\mathrm{f}}(\mathcal{Q}))$ **and** $\mathcal{U}(\mathcal{P}^\sharp_{\mathrm{f}}(\mathcal{Q}))$ are countable spaces with a countable basis;

$\mathcal{U}(\mathcal{P}^\flat(\mathcal{Q}))$ **and** $\mathcal{U}(\mathcal{P}^\sharp(\mathcal{Q}))$ are uncountable spaces and we described their topology using an uncountable basis.

Countable second countable spaces

Dorais introduced a framework for dealing with countable second countable spaces.

Definition (RCA_0)

A *countable second-countable space* consists of a set X, a sequence $(U_i)_{i \in I}$ of subsets of X, and a function $k : X \times I \times I \to I$ such that

- if $x \in X$, then $x \in U_i$ for some $i \in I$;
- if $x \in U_i \cap U_j$, then $x \in U_{k(x,i,j)} \subseteq U_i \cap U_j$.

Coding open sets and expressing compactness

Every function $h : \mathbb{N} \to \mathcal{P}_f(I)$ codes the *open* set $G_h = \bigcup_{n\in\mathbb{N}} \bigcup_{i\in h(n)} U_i$.

Definition (RCA_0)

The open set G_h is *compact* if for every $f : \mathbb{N} \to \mathcal{P}_f(I)$ with $G_h \subseteq \bigcup_{n\in\mathbb{N}} \bigcup_{i\in f(n)} U_i$, there exists N such that $G_h \subseteq \bigcup_{n<N} \bigcup_{i\in f(n)} U_i$.

Equivalent definitions of Noetherian are equivalent

Lemma (RCA$_0$)

For a countable second-countable space $(X, (U_i)_{i\in I}, k)$, *the following statements are equivalent:*

- **(i)** *every open set is compact;*
- **(ii)** *for every open set* G_h, *there exists* N *such that* $G_h = \bigcup_{n<N} \bigcup_{i\in h(n)} U_i$;
- **(iii)** *for every sequence* $(G_n)_{n\in\mathbb{N}}$ *of open sets such that* $\forall n\, G_n \subseteq G_{n+1}$, *there exists* N *such that* $\forall n > N\, G_n = G_N$;
- **(iv)** *for every sequence* $(F_n)_{n\in\mathbb{N}}$ *of closed sets such that* $\forall n\, F_n \supseteq F_{n+1}$, *there exists* N *such that* $\forall n > N\, F_n = F_N$.

Definition (RCA$_0$)

A countable second-countable space is *Noetherian* if it satisfies any of the equivalent conditions above.

Coding the Alexandroff and upper topologies

Definition (RCA_0)

Let $\mathcal{Q}$ be a quasi-order.

- A base for the Alexandroff topology on $\mathcal{Q}$ is given by $(U_q)_{q \in Q}$, where $U_q = q\uparrow$ for each $q \in Q$, and $k(q,p,r) = q$.
  Let $\mathcal{A}(\mathcal{Q})$ denote the countable second-countable space $(Q, (U_q)_{q \in Q}, k)$.
- A base for the upper topology on $\mathcal{Q}$ is given by $(V_{\mathbf{i}})_{\mathbf{i} \in \mathcal{P}_{\mathrm{f}}(Q)}$, where $V_{\mathbf{i}} = Q \setminus (\mathbf{i}\downarrow)$ for each $\mathbf{i} \in \mathcal{P}_{\mathrm{f}}(Q)$, and $\ell(q, \mathbf{i}, \mathbf{j}) = \mathbf{i} \cup \mathbf{j}$.
  Let $\mathcal{U}(\mathcal{Q})$ denote the countable second-countable space $(Q, (V_{\mathbf{i}})_{\mathbf{i} \in \mathcal{P}_{\mathrm{f}}(Q)}, \ell)$.

Basic facts

Lemma (RCA_0)

Let $\mathcal{Q}$ be a quasi-order.

(i) *If $\mathcal{A}(\mathcal{Q})$ Noetherian, then $\mathcal{U}(\mathcal{Q})$ Noetherian.*

(ii) *$\mathcal{Q}$ is wqo if and only if $\mathcal{A}(\mathcal{Q})$ is Noetherian.*

Corollary (ACA_0)

If $\mathcal{Q}$ is wqo then $\mathcal{A}(\mathcal{P}_f^\flat(\mathcal{Q}))$ and $\mathcal{U}(\mathcal{P}_f^\flat(\mathcal{Q}))$ are Noetherian.

We can also express "if $\mathcal{Q}$ is wqo then $\mathcal{U}(\mathcal{P}_f^\sharp(\mathcal{Q}))$ is Noetherian" in RCA_0.

$\mathcal{U}(\mathcal{P}^\flat(\mathcal{Q}))$ and $\mathcal{U}(\mathcal{P}^\sharp(\mathcal{Q}))$ are second countable

$\mathcal{U}(\mathcal{P}^\flat(\mathcal{Q}))$ and $\mathcal{U}(\mathcal{P}^\sharp(\mathcal{Q}))$ are spaces with uncountably many points. Moreover we described their topology using uncountable basis.

However both spaces have (non-obvious) countable basis.

Fact

The sets of the form $\{\, Q \setminus (q\uparrow) \mid q \in \mathbf{i} \,\}\downarrow^\flat$, *where* $\mathbf{i} \in \mathcal{P}_\mathrm{f}(Q)$, *are a basis for the closed sets of the topology of* $\mathcal{U}(\mathcal{P}^\flat(\mathcal{Q}))$.

Fact

The sets of the form $\{\, \{q\} \mid q \in \mathbf{i} \,\}\downarrow^\sharp$, *where* $\mathbf{i} \in \mathcal{P}_\mathrm{f}(Q)$, *are a basis for the closed sets of the topology of* $\mathcal{U}(\mathcal{P}^\sharp(\mathcal{Q}))$.

Where is second countability provable?

Lemma (RCA_0)

The following are equivalent:

(i) ACA_0;

(ii) *If* $\mathcal{Q}$ *is a quasi-order and* $E \subseteq Q$*, then* $\{E\}\downarrow^\flat$ *is a countable intersection of sets of the form* $\{\, Q \setminus (q\uparrow) \mid q \in \mathbf{i} \,\}\downarrow^\flat$*, with* $\mathbf{i} \in \mathcal{P}_\mathrm{f}(Q)$*;*

(iii) *the same statement when* $\mathcal{Q}$ *is a well order.*

Lemma (RCA_0)

If $\mathcal{Q}$ *is a quasi-order and* $E \subseteq Q$*, then* $\{E\}\downarrow^\sharp$ *is a countable intersection of sets of the form* $\{\, \{q\} \mid q \in \mathbf{i} \,\}\downarrow^\sharp$*, with* $\mathbf{i} \in \mathcal{P}_\mathrm{f}(Q)$*.*

A scheme for representing uncountable second-countable spaces

A *second-countable space* is coded by a set $I \subseteq \mathbb{N}$ and formulas $\varphi(X)$, $\Psi_=(X,Y)$, and $\Psi_\in(X,n)$

I is the set of codes for open sets

$\varphi(X)$ means "X codes a point"

$\Psi_=(X,Y)$ means "X and Y code the same point"

$\Psi_\in(X,i)$ means "the point coded by X belongs to the open set coded by $i \in I$"

We ask that

- if $\varphi(X)$, then $\Psi_\in(X,i)$ for some $i \in I$;
- if $\varphi(X)$, $\Psi_\in(X,i)$, and $\Psi_\in(X,j)$ for some $i,j \in I$, then there exists $k \in I$ such that $\Psi_\in(X,k)$ and $\forall Y[\Psi_\in(Y,k) \implies (\Psi_\in(Y,i) \wedge \Psi_\in(Y,j))]$;
- if $\varphi(X)$, $\varphi(Y)$, $\Psi_\in(X,i)$ for some $i \in I$, and $\Psi_=(X,Y)$, then $\Psi_\in(Y,i)$.

Old codings of spaces fit in this scheme

When we code a complete separable metric space (A, d) using a countable dense set A, we let $I = A \times \mathbb{Q}^+$ and then set

- $\varphi(X) \overset{\text{def}}{=}$ "X is a rapidly converging Cauchy sequence of points in A"
- $\Psi_=(X, Y) \overset{\text{def}}{=}$ "the distances between the points of the sequences X and Y go to 0 fast enough"
- $\Psi_\in(X, (a, q)) \overset{\text{def}}{=}$ "the distance between the point coded by X and $a \in A$ is less than $q \in \mathbb{Q}^+$"

Also, Mummert's MF spaces (second-countable T_1 spaces with the strong Choquet property) can be accomodated by our scheme.

The codings for $\mathcal{U}(\mathcal{P}^\flat(\mathcal{Q}))$ and $\mathcal{U}(\mathcal{P}^\sharp(\mathcal{Q}))$

Definition (RCA_0)

Let $\mathcal{Q}$ be a quasi-order.
The second-countable space $\mathcal{U}(\mathcal{P}^\flat(\mathcal{Q}))$ is coded by the set $I = \mathcal{P}_{\mathrm{f}}(Q)$ and the formulas:

- $\varphi(X) \stackrel{\text{def}}{=} X \subseteq Q$;
- $\Psi_=(X, Y) \stackrel{\text{def}}{=} X = Y$;
- $\Psi_\in(X, \mathbf{i}) \stackrel{\text{def}}{=} \mathbf{i} \subseteq X\downarrow$.

The second-countable space $\mathcal{U}(\mathcal{P}^\sharp(\mathcal{Q}))$ is coded by the set $I = \mathcal{P}_{\mathrm{f}}(Q)$ and the formulas:

- $\varphi(X) \stackrel{\text{def}}{=} X \subseteq Q$;
- $\Psi_=(X, Y) \stackrel{\text{def}}{=} X = Y$;
- $\Psi_\in(X, \mathbf{i}) \stackrel{\text{def}}{=} \mathbf{i} \cap X\uparrow = \emptyset$.

Relations between topologies

Using these codings we can formalize the statements "$\mathcal{U}(\mathcal{P}^\flat(\mathcal{Q}))$ is Noetherian" and "$\mathcal{U}(\mathcal{P}^\sharp(\mathcal{Q}))$ is Noetherian" (the equivalence of the various definitions is provable in RCA_0).

In general, $\mathcal{U}(\mathcal{P}^\flat_{\mathrm{f}}(\mathcal{Q}))$ is strictly coarser than the subspace topology on $\mathcal{P}_{\mathrm{f}}(\mathcal{Q})$ induced by $\mathcal{U}(\mathcal{P}^\flat(\mathcal{Q}))$.

However, $\mathcal{U}(\mathcal{P}^\sharp_{\mathrm{f}}(\mathcal{Q}))$ is the subspace topology on $\mathcal{P}_{\mathrm{f}}(\mathcal{Q})$ induced by $\mathcal{U}(\mathcal{P}^\sharp(\mathcal{Q}))$.

Theorem (RCA_0)

Let $\mathcal{Q}$ be a quasi-order.

1. *If $\mathcal{U}(\mathcal{P}^\flat(\mathcal{Q}))$ is Noetherian, then $\mathcal{U}(\mathcal{P}^\flat_{\mathrm{f}}(\mathcal{Q}))$ is Noetherian.*
2. *If $\mathcal{U}(\mathcal{P}^\sharp(\mathcal{Q}))$ is Noetherian, then $\mathcal{U}(\mathcal{P}^\sharp_{\mathrm{f}}(\mathcal{Q}))$ is Noetherian.*

This is not entirely trivial because the codings are different!

Proving Goubault-Larrecq's theorems

Theorem (ACA_0)

If $\mathcal{Q}$ is wqo then $\mathcal{U}(\mathcal{P}^\flat(\mathcal{Q}))$ and $\mathcal{U}(\mathcal{P}^\sharp(\mathcal{Q}))$ are Noetherian.

Goubault-Larrecq's original proofs are category-theoretic.
We need to use completely different, more elementary, arguments.

Corollary (ACA_0)

If $\mathcal{Q}$ is wqo then $\mathcal{U}(\mathcal{P}^\flat_{\mathrm{f}}(\mathcal{Q}))$ and $\mathcal{U}(\mathcal{P}^\sharp_{\mathrm{f}}(\mathcal{Q}))$ are Noetherian.

The reversals

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The strategy for reversals

We want to show that

if $\mathcal{Q}$ is a wqo, then $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^{\star}(\mathcal{Q}))$ is Noetherian

implies ACA_0 over RCA_0 (where $\star \in \{\flat, \sharp\}$).

Our strategy is to produce, given an injective $f : \mathbb{N} \to \mathbb{N}$, a f-computable $\mathcal{Q}$ such that RCA_0 proves:

- $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^{\star}(\mathcal{Q}))$ is not Noetherian;
- if g is a bad sequence in $\mathcal{Q}$, then $g \oplus f$ computes $\mathrm{ran}(f)$.

True and false stages of f

$f : \mathbb{N} \to \mathbb{N}$ is injective

- n is f-*true* if $\forall k > n\, f(n) < f(k)$;
- n is f-*true at stage* s if $n < s$ and $\forall k\, (n < k \leq s \implies f(n) < f(k))$.

Otherwise n is false (at stage s).

If g is an injective sequence of true numbers, then $\operatorname{ran}(f) \leq_T g \oplus f$ because we may assume that g is strictly increasing and then

$$k \in \operatorname{ran}(f) \iff \exists n \leq g(k)\, f(n) = k.$$

A pseudo well order

$f : \mathbb{N} \to \mathbb{N}$ is injective

The prototype of a construction using true and false stages produces a linear order $\mathcal{L}$ such that

- $\mathcal{L}$ has order type $\omega + \omega^*$;
- the ω part of $\mathcal{L}$ consists of the f-false stages and is Σ^0_1 in f;
- the ω^* part of $\mathcal{L}$ consists of the f-true stages and is Π^0_1 in f.

Thus, if we know that $\mathcal{L}$ is not a well order then we can compute $\operatorname{ran}(f)$.

$\mathcal{L}$ is defined recursively: we put the new element s below the n's that are f-true at stage s and above the n's that are f-false at stage s.

Generalizing the construction

$f : \mathbb{N} \to \mathbb{N}$ is injective

We generalize the previous construction: rather then adding one element, at each stage we add a finite partial order $\mathcal{R}$ with a designated point x.

By controlling how the s-th copy of $\mathcal{R}$ sits into the construction (depending on the n's that are f-true and f-false at stage s) we define a partial order $\Xi_f(\mathcal{R}, x)$ so that

Lemma (RCA_0)

If $\Xi_f(\mathcal{R}, x)$ *is not a wqo then* $\operatorname{ran}(f)$ *exists.*

The reversals

$f : \mathbb{N} \to \mathbb{N}$ is injective

Making appropriate choices of $\mathcal{R}$ and x we build $\mathcal{Q} = \Xi_f(\mathcal{R}, x)$ such that $\mathcal{U}(\mathcal{P}_\mathrm{f}^\flat(\mathcal{Q}))$ is not Noetherian and obtain

Theorem (RCA$_0$)

The statement "if $\mathcal{Q}$ is wqo then $\mathcal{U}(\mathcal{P}_\mathrm{f}^\flat(\mathcal{Q}))$ is Noetherian" implies ACA$_0$.

Using a different $\mathcal{R}$ we get

Theorem (RCA$_0$)

The statement "if $\mathcal{Q}$ is wqo then $\mathcal{U}(\mathcal{P}_\mathrm{f}^\sharp(\mathcal{Q}))$ is Noetherian" implies ACA$_0$.

The main result

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Summing up: the reverse mathematics of Goubault-Larrecq's theorems

Main Theorem (RCA_0)

The following are equivalent:

(i) ACA_0;

(ii) *if* $\mathcal{Q}$ *is wqo then* $\mathcal{A}(\mathcal{P}_{\mathrm{f}}^{\flat}(\mathcal{Q}))$ *is Noetherian;*

(iii) *if* $\mathcal{Q}$ *is wqo then* $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^{\flat}(\mathcal{Q}))$ *is Noetherian;*

(iv) *if* $\mathcal{Q}$ *is wqo then* $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^{\sharp}(\mathcal{Q}))$ *is Noetherian;*

(v) *if* $\mathcal{Q}$ *is wqo then* $\mathcal{U}(\mathcal{P}^{\flat}(\mathcal{Q}))$ *is Noetherian;*

(vi) *if* $\mathcal{Q}$ *is wqo then* $\mathcal{U}(\mathcal{P}^{\sharp}(\mathcal{Q}))$ *is Noetherian.*

Finer analysis?

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Π^1_2 statements

Many theorems studied in reverse mathematics are $\mathbf{\Pi}^1_2$ statements of the form

$$\forall X (\Phi(X) \implies \exists Y \, \Psi(X, Y))$$

where Φ and Ψ are arithmetical.

In this situation we often say that an X such that $\Phi(X)$ is a problem, and a Y satisfying $\Psi(X, Y)$ is a solution to the problem.

We look at the multi-valued map assigning to a problem the set of its solutions.
We compare these multi-valued maps using (strong) Weihrauch reducibility and/or (strong) reducibility.

These reductions lead to a finer analysis of the strength of the statements.

Goubault-Larrecq's theorems as Π^1_2 statements

Goubault-Larrecq's theorems are indeed $\mathbf{\Pi}^1_2$ statements, but they are of the following form:

$$\forall X (\forall Z\, \Phi(X, Z) \implies \forall Y\, \Psi(X, Y))$$

with Φ and Ψ arithmetical.

In fact both "$\mathcal{Q}$ is wqo" and "$\mathcal{U}(\mathcal{Q})$ is Noetherian" are $\mathbf{\Pi}^1_1$.

These statements do not fit nicely into the problem/solution pattern.

We can rewrite them as

$$\forall X\, \forall Y (\neg\Psi(X, Y) \implies \exists Z\, \neg\Phi(X, Z)).$$

A problem is a pair consisting of a quasi-order $\mathcal{Q}$ and a witness to the fact that $\mathcal{U}(\mathcal{P}^\flat_\mathrm{f}(\mathcal{Q}))$ is not Noetherian.
Its solutions are the bad sequences in $\mathcal{Q}$.

Which is the real form of Goubault-Larrecq's theorems?

In fact our proofs of both directions of the reverse mathematics results actually consider statements such as

if $\mathcal{U}(\mathcal{P}_{\mathrm{f}}^{\flat}(\mathcal{Q}))$ *is not Noetherian then* $\mathcal{Q}$ *is not wqo*

Thank you for your attention!