Partial Functions and Domination

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Domination for Partial Functions

Definition

Let $f, g \subset \omega^\omega$ be partial functions. Then g dominates f if for all sufficiently large n, if $f(n)$ is defined, then $f(n) \leq g(m)$ for some $m \leq n$ such that $g(m)$ is defined.

Definition

Let $A \subseteq \omega$. Then A is *pdominant* if there is an e such that Φ_e^A dominates every partial recursive function.

Problem: Study the recursion-theoretic properties of pdominant sets.

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History and Motivation

- For total functions, the corresponding notion of domination is well investigated.
- (Martin 1967) An r.e. set A is high (i.e. $A' \equiv_T \emptyset''$) if and only if there is an e such that Φ_e^A is total and for each total recursive f, $\Phi_e^A(n) \geq f(n)$ for all sufficiently large n.
- Functions dominating partial recursive functions (called "self-generating functions") occur naturally in the construction of a nonstandard model of SRT_2^2 in which RT_2^2 fails. Controlling their growth rates is a major issue.
- It leads to the introduction of the BME_k ($k < \omega$) principle (Chong, Slaman and Yang (2014)).

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- Over $\mathrm{RCA}_0 + B\Sigma_2$, BME_1 is equivalent to $P\Sigma_1$.
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Π^0_1 Class and pDomination

Theorem

1. *There is a nontrivial Π^0_1 class with no pdominant members.*
2. *There is a Π^0_1 class with only pdominant members.*

Proof.
(1). Construct a partial recursive function and let the Π^0_1 class be the collection of all its total extensions.
(2). There is a Π^0_1 class whose only nonrecursive member has complete Turing degree.

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Genericity and pDomination

An *extension function* is a partial function h mapping binary strings to binary strings such that if $h(\sigma)$ is defined, then $\sigma \subset h(\sigma)$.

A is 1-generic if it meets every partial recursive extension function.

A is *weakly* 2-generic if it meets every partial $\emptyset'$-recursive extension function.

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1. *There is a 1-generic set that is pdominant.*
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