Cardinal invariants of density

Dilip Raghavan

National University of Singapore

Computability Theory and Foundations of Mathematics 2015, Tokyo Institute of Technology, Tokyo, Japan
September 8, 2015
Basic definitions

Definition

$I \subseteq \mathcal{P}(\omega)$ is an \textbf{ideal on} $\omega$ if

1. $I$ is closed under subsets and finite unions.
2. Every finite subset of $\omega$ belongs to $I$.
3. $\omega \notin I$.

In this talk I am primarily interested in $I$ that are definable.
The P-ideals form a special class.

**Definition**

An ideal $\mathcal{I}$ on $\omega$ is called a **P-ideal** if $\mathcal{I}$ is countably directed mod finite. In other words, if $\{a_n : n \in \omega\} \subseteq \mathcal{I}$, then there exists $a \in \mathcal{I}$ such that $\forall n \in \omega \ [a_n \subseteq^* a]$.

Here $a \subseteq^* b$ means $a \setminus b$ is finite.
The P-ideals form a special class.

**Definition**

An ideal $\mathcal{I}$ on $\omega$ is called a **P-ideal** if $\mathcal{I}$ is countably directed mod finite. In other words, if $\{a_n : n \in \omega\} \subseteq \mathcal{I}$, then there exists $a \in \mathcal{I}$ such that $\forall n \in \omega \ [a_n \subseteq^* a]$.

Here $a \subseteq^* b$ means $a \setminus b$ is finite.

Being a P-ideal has a strong influence on the structure of an ideal $\mathcal{I}$.

It also influences the possible definable complexity of $\mathcal{I}$. 
• $\mathcal{P}(\omega)$ is a Polish space with the usual Cantor topology.
• Sets of the form $\{X \subseteq \omega : n \in X\}$ and $\{X \subseteq \omega : n \notin X\}$ form a sub-basis.
• We can talk about the complexity of $\mathcal{I}$ in the descriptive sense.
• The simplest are the $\mathcal{F}_\sigma$ ideals.
These have a characterization in terms of sub-measures:

**Definition**

A function $\phi : \mathcal{P}(\omega) \to [0, \infty]$ is called a **sub-measure** if

1. $\phi(0) = 0$ and $\phi(\{n\}) < \infty$, for every $n \in \omega$;
2. $X \subseteq Y \Rightarrow \phi(X) \leq \phi(Y)$;
3. $\phi(X \cup Y) \leq \phi(X) + \phi(Y)$;

**Definition**

A sub-measure $\phi$ is **lower semi-continuous (lsc)** if for any $X \subseteq \omega$,

$$\phi(X) = \lim_{n \to \infty} \phi(X \cap n).$$
Fact (Mazur)

An ideal \( \mathcal{I} \) on \( \omega \) is \( F_\sigma \) iff \( \mathcal{I} = \text{Fin}(\phi) = \{X \subseteq \omega : \phi(X) < \infty\} \).

Example

\( \mathcal{I} \frac{1}{n} \) is the ideal of **summable sets**. That is

\[
\mathcal{I} \frac{1}{n} = \left\{ X \subseteq \omega : \sum_{n \in X} \frac{1}{n} < \infty \right\}
\]

- \( \mathcal{I} \frac{1}{n} \) is actually a P-ideal.
- The sub-measure here is just \( \phi(X) = \sum_{n \in X} \frac{1}{n} \).
- Can replace \( \frac{1}{n} \) by any divergent series (the ideals are quite different though!).
Example

\( \mathcal{ED} \) is the ideal on \( \omega \times \omega \) generated by the vertical columns and graphs of functions. That is \( \mathcal{ED} = \)

\[ \{ X \subseteq \omega \times \omega : \exists k, l \in \omega \forall n > k \left[ |\{ m \in \omega : \langle n, m \rangle \in X \}| \leq l \right] \} \]

- This is an \( F_\sigma \) ideal which is not P.
Moving up the complexity hierarchy, it turns out that every analytic P-ideal is $F_{\sigma\delta}$.

So at least for P-ideals, there is nothing between $F_{\sigma\delta}$ and $\Pi_1^1$. 
Moving up the complexity hierarchy, it turns out that every analytic P-ideal is $F_{\sigma\delta}$.

So at least for P-ideals, there is nothing between $F_{\sigma\delta}$ and $\Pi_1^1$.

**Theorem (Solecki)**

Let $I$ be an ideal on $\omega$.

1. $I$ is an analytic P-ideal iff there exists a lower semi-continuous sub-measure $\phi$ such that $I = \text{Exh}(\phi) = \{X \subseteq \omega : \lim_{n \to \infty} \phi(X \setminus n) = 0\}$.

2. $I$ is an $F_{\sigma\delta}$ P-ideal iff there exists a lower semi-continuous sub-measure $\phi$ such that $I = \text{Fin}(\phi) = \text{Exh}(\phi)$.

Exh($\phi$) is always an $F_{\sigma\delta}$ P-ideal.
A set $A \subseteq \omega$ is said to have asymptotic density $0$ if \( \lim_{n \to \infty} \frac{|A \cap n|}{n} = 0 \).

\[ Z_0 = \left\{ A \subseteq \omega : \lim_{n \to \infty} \frac{|A \cap n|}{n} = 0 \right\} . \]

- This an \( F_{\sigma\delta} \) P-ideal.
- Suppose \( \{a_n : n \in \omega\} \subseteq Z_0 \).
- WLOG they are pairwise disjoint.
Let $b_n = \bigcup_{m \leq n} a_m$ and let $k_n$ be minimal such that for all $k \geq k_n$, 
\[
\frac{|b_n \cap k|}{k} \leq 2^{-n}.
\]
Let $a = \bigcup_{n \in \omega} (a_n \setminus k_n)$.

This set $a$ works.
Three basic invariants

- Cardinals invariants are cardinal between $\aleph_1$ and $c = 2^{\aleph_0}$.
- They identify places where basic diagonalization arguments first fail.
Three basic invariants

Cardinals invariants are cardinal between $\aleph_1$ and $c = 2^{\aleph_0}$.
They identify places where basic diagonalization arguments first fail.

**Definition**

For $f, g \in \omega^\omega$, $f <^* g$ means that $|\{n \in \omega : g(n) \leq f(n)\}| < \omega$. A set $F \subseteq \omega^\omega$ is said to be **unbounded** if there does not exist $g \in \omega^\omega$ such that $\forall f \in F [f <^* g]$. A set $F \subseteq \omega^\omega$ is said to be **dominating or cofinal** if $\forall f \in \omega^\omega \exists g \in F [f <^* g]$. 
Three basic invariants

- Cardinals invariants are cardinal between $\aleph_1$ and $\mathfrak{c} = 2^{\aleph_0}$.
- They identify places where basic diagonalization arguments first fail.

**Definition**

For $f, g \in \omega^\omega$, $f <^* g$ means that $|\{n \in \omega : g(n) \leq f(n)\}| < \omega$. A set $F \subseteq \omega^\omega$ is said to be **unbounded** if there does not exist $g \in \omega^\omega$ such that $\forall f \in F [f <^* g]$. A set $F \subseteq \omega^\omega$ is said to be **dominating or cofinal** if $\forall f \in \omega^\omega \exists g \in F [f <^* g]$.

**Definition**

For $a, b \in \mathcal{P}(\omega)$ we say that $a$ **splits** $b$ if both $b \cap a$ and $b \cap (\omega \setminus a)$ are infinite. A family $F \subseteq \mathcal{P}(\omega)$ is called a **splitting family** if $\forall b \in [\omega]^{\omega} \exists a \in F [a \text{ splits } b]$. 
We define the cardinal invariants $b$, $d$, and $s$ as follows:

$$b = \min\{|F| : F \subseteq \omega^\omega \land F \text{ is unbounded}\};$$
$$d = \min\{|F| : F \subseteq \omega^\omega \land F \text{ is dominating}\};$$
$$s = \min\{|F| : F \subseteq \mathcal{P}(\omega) \land F \text{ is a splitting family}\}.$$  

Fact

$\aleph_1 \leq \max\{b, s\} \leq d \leq c$.

- This is all that can be proved in ZFC.
We consider cardinal invariants associated with analytic P-ideals.

Two possibilities: invariants associated with the quotient $\mathcal{P}(\omega)/\mathcal{I}$ and cardinals associated with $\mathcal{I}$ itself.

Former is similar to $\mathcal{P}(\omega)/\text{FIN}$.

The latter involves possibilities that don’t make sense for FIN because FIN is not a tall ideal.

**Definition**

*Recall that an ideal $\mathcal{I}$ on $\omega$ is **tall** if it has the property that*

$$\forall a \in [\omega]^\omega \exists b \in [a]^\omega [b \in \mathcal{I}].$$
• When $\mathcal{I}$ is a tall P-ideal, we can define invariants associated with $\mathcal{I}$ that don’t make sense for FIN.

• There are many interesting open problems about invariants associated with $\mathcal{P}(\omega)/\mathcal{I}$ (not our topic for today, but . . . ).
When $\mathcal{I}$ is a tall P-ideal, we can define invariants associated with $\mathcal{I}$ that don’t make sense for FIN.

There are many interesting open problems about invariants associated with $\mathcal{P}(\omega)/\mathcal{I}$ (not our topic for today, but . . .).

**Definition**

A family $F \subseteq \mathcal{P}(\omega)$ is **splitting for** $\mathcal{P}(\omega)/\mathcal{I}_{\frac{1}{n}}$ if

$$\forall b \in \mathcal{I}_{\frac{1}{n}}^{+} \exists a \in F \left[ b \cap a \in \mathcal{I}_{\frac{1}{n}}^{+} \wedge b \cap (\omega \setminus a) \in \mathcal{I}_{\frac{1}{n}}^{+} \right].$$
When $\mathcal{I}$ is a tall P-ideal, we can define invariants associated with $\mathcal{I}$ that don’t make sense for FIN.

There are many interesting open problems about invariants associated with $\mathcal{P}(\omega)/\mathcal{I}$ (not our topic for today, but . . . ).

**Definition**

A family $F \subseteq \mathcal{P}(\omega)$ is **splitting for** $\mathcal{P}(\omega)/\mathcal{I}_{1/n}$ if

$$\forall b \in \mathcal{I}_{1/n}^+ \exists a \in F \left[ b \cap a \in \mathcal{I}_{1/n}^+ \land b \cap (\omega \setminus a) \in \mathcal{I}_{1/n}^+ \right].$$

**Definition**

The analogue of $s$ for $\mathcal{P}(\omega)/\mathcal{I}_{1/n}$ is:

$$s_{1/n} = \min \left\{|F| : F \subseteq \mathcal{P}(\omega) \text{ is splitting for } \mathcal{P}(\omega)/\mathcal{I}_{1/n}\right\}.$$
Theorem (Brendle)

It is consistent to have $\mathfrak{s} \frac{1}{n} < \mathfrak{s}$.

Question

Is $\mathfrak{s} < \mathfrak{s} \frac{1}{n}$ consistent?

Question

Is $\mathfrak{h} < \mathfrak{h} \frac{1}{n}$ consistent?
When $\mathcal{I}$ is a tall P-ideal on $\omega$ you can define the following:

\[
\text{add}^*(\mathcal{I}) = \min\{ |\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \land \forall b \in \mathcal{I} \exists a \in \mathcal{F} \; [a \not\in^* b] \}, \\
\text{cov}^*(\mathcal{I}) = \min\{ |\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \land \forall a \in [\omega]^{\omega} \exists b \in \mathcal{F} \; [|a \cap b| = \omega] \}, \\
\text{cof}^*(\mathcal{I}) = \min\{ |\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \land \forall b \in \mathcal{I} \exists a \in \mathcal{F} \; [b \subseteq^* a] \}, \\
\text{non}^*(\mathcal{I}) = \min\{ |\mathcal{F}| : \mathcal{F} \subseteq [\omega]^{\omega} \land \forall b \in \mathcal{I} \exists a \in \mathcal{F} \; [|a \cap b| < \omega] \}.
\]
Definition

When $\mathcal{I}$ is a tall $P$-ideal on $\omega$ you can define the following:

$$ add^*(\mathcal{I}) = \min\{|F| : F \subseteq \mathcal{I} \land \forall b \in \mathcal{I} \exists a \in F \left[ a \not\subseteq^* b \right] \}, $$

$$ cov^*(\mathcal{I}) = \min\{|F| : F \subseteq \mathcal{I} \land \forall a \in [\omega]^{\omega} \exists b \in F \left[ |a \cap b| = \omega \right] \}, $$

$$ cof^*(\mathcal{I}) = \min\{|F| : F \subseteq \mathcal{I} \land \forall b \in \mathcal{I} \exists a \in F \left[ b \subseteq^* a \right] \}, $$

$$ non^*(\mathcal{I}) = \min\{|F| : F \subseteq [\omega]^{\omega} \land \forall b \in \mathcal{I} \exists a \in F \left[ |a \cap b| < \omega \right] \}. $$

- If $\mathcal{I}$ were not a $P$-ideal, $add^*(\mathcal{I})$ would be $\omega$.
- If $\mathcal{I}$ were not tall, then $cov^*(\mathcal{I})$ would be undefined, and $non^*(\mathcal{I})$ would be $1$. 
These invariants were investigated by Hernández-Hernández and Hrušák [2] and also by Brendle and Shelah [1].

Terminology is based on analogy with the following definitions which make sense for any ideal whatsoever.

**Definition**

Let $\mathcal{I}$ be any ideal on a set $X$. Define

\[
\text{add}(\mathcal{I}) = \min \{ |\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \land \bigcup \mathcal{F} \notin \mathcal{I} \}, \\
\text{cov}(\mathcal{I}) = \min \{ |\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \land \bigcup \mathcal{F} = X \}, \\
\text{cof}(\mathcal{I}) = \min \{ |\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \land \forall B \in \mathcal{I} \exists A \in \mathcal{F} \ [B \subseteq A] \}, \\
\text{non}(\mathcal{I}) = \{ |Y| : Y \subseteq X \land Y \notin \mathcal{I} \}.
\]
These invariants were investigated by Hernández-Hernández and Hrušák [2] and also by Brendle and Shelah [1].

Terminology is based on analogy with the following definitions which make sense for any ideal whatsoever.

**Definition**

*Let $\mathcal{I}$ be any ideal on a set $X$. Define*

\[
\begin{align*}
\text{add}(\mathcal{I}) &= \min \{ |\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \land \bigcup \mathcal{F} \not\in \mathcal{I} \}, \\
\text{cov}(\mathcal{I}) &= \min \{ |\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \land \bigcup \mathcal{F} = X \}, \\
\text{cof}(\mathcal{I}) &= \min \{ |\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \land \forall B \in \mathcal{I} \exists A \in \mathcal{F} \ [B \subseteq A] \}, \\
\text{non}(\mathcal{I}) &= \{ |Y| : Y \subseteq X \land Y \not\in \mathcal{I} \}.
\end{align*}
\]

*add*(\mathcal{I}) and *cof*(\mathcal{I}) are duals. So are *cov*(\mathcal{I}) and *non*(\mathcal{I}).
For each $a \in \mathcal{P}(\omega)$, let $\hat{a} = \{b \subseteq \omega : |a \cap b| = \omega\}$.

For each $a \in \mathcal{P}(\omega)$, let $\hat{a} = \{b \subseteq \omega : |a \cap b| = \omega\}$.

For a tall ideal $\mathcal{I}$, $\hat{\mathcal{I}} = \{X \subseteq \mathcal{P}(\omega) : \exists a \in \mathcal{I} [X \subseteq \hat{a}]\}$ is an ideal on $\mathcal{P}(\omega)$ generated by Borel sets.
For each $a \in \mathcal{P}(\omega)$, let \( \hat{a} = \{ b \subseteq \omega : |a \cap b| = \omega \} \).

For each $a \in \mathcal{P}(\omega)$, let \( \hat{a} = \{ b \subseteq \omega : |a \cap b| = \omega \} \).

For a tall ideal $\mathcal{I}$, \( \hat{\mathcal{I}} = \{ X \subseteq \mathcal{P}(\omega) : \exists a \in \mathcal{I} [X \subseteq \hat{a}] \} \) is an ideal on $\mathcal{P}(\omega)$ generated by Borel sets.

$\mathcal{I}$ is a P-ideal iff $\hat{\mathcal{I}}$ is a $\sigma$-ideal.

$\text{add}(\hat{\mathcal{I}}) = \text{add}^{*}(\mathcal{I})$, $\text{cov}(\hat{\mathcal{I}}) = \text{cov}^{*}(\mathcal{I})$, $\text{cof}(\hat{\mathcal{I}}) = \text{cof}^{*}(\mathcal{I})$, $\text{non}(\hat{\mathcal{I}}) = \text{non}^{*}(\mathcal{I})$ hold.
The Tukey and the Katětov orderings are relevant to these invariants.

**Definition**
Let $I$ and $J$ be ideals on $\omega$. Recall that $I$ is Katětov below $J$ or $I \leq_K J$ if there is a function $f : \omega \to \omega$ such that $\forall a \in I \left[f^{-1}(a) \in J\right]$.

**Definition**
We say that $\langle I, \subseteq^* \rangle$ is Tukey below $\langle J, \subseteq^* \rangle$ and we write $I \leq_T^* J$ if there is a map $\varphi : I \to J$ such that if $X \subseteq I$ any set that does not have an upper bound in the partial order $\langle I, \subseteq^* \rangle$, then $\varphi''X$ does not have an upper bound in the partial order $\langle J, \subseteq^* \rangle$. 
The Tukey and the Katětov orderings are relevant to these invariants.

**Definition**

Let $I$ and $J$ be ideals on $\omega$. Recall that $I$ is **Katětov below** $J$ or $I \leq_K J$ if there is a function $f : \omega \to \omega$ such that $\forall a \in I \left[ f^{-1}(a) \in J \right]$.

**Definition**

We say that $\langle I, \subseteq^* \rangle$ is **Tukey below** $\langle J, \subseteq^* \rangle$ and we write $I \leq_T^* J$ if there is a map $\varphi : I \to J$ such that if $X \subseteq I$ any set that does not have an upper bound in the partial order $\langle I, \subseteq^* \rangle$, then $\varphi''X$ does not have an upper bound in the partial order $\langle J, \subseteq^* \rangle$.

- $I \leq_K J$ implies both that $\text{cov}^*(I) \geq \text{cov}^*(J)$ and that $\text{non}^*(I) \leq \text{non}^*(J)$.
- If $I \leq_T^* J$, then $\text{add}^*(I) \geq \text{add}^*(J)$ and $\text{cof}^*(I) \leq \text{cof}^*(J)$. 
Summary of some known results:

Fact

Let $\mathcal{I}$ be a tall P-ideal on $\omega$.

1. $\aleph_1 \leq \text{add}^*(\mathcal{I}) \leq \min\{\text{non}^*(\mathcal{I}), \text{cov}^*(\mathcal{I})\} \leq \max\{\text{non}^*(\mathcal{I}), \text{cov}^*(\mathcal{I})\} \leq \text{cof}^*(\mathcal{I}) \leq c$.

2. $p \leq \text{cov}^*(\mathcal{I})$. 
The following hold:

1. \[\text{add}^*\left(\mathcal{I}_{\frac{1}{n}}\right) = \text{add}(\mathcal{N}).\]

2. (Todorcevic) For every analytic P-ideal \(\mathcal{I}\), \(0 \times \text{FIN} \leq^*_T \mathcal{I} \leq^*_T \mathcal{I}_{\frac{1}{n}}\).
   Therefore \(\text{add}(\mathcal{N}) \leq \text{add}^*(\mathcal{I}) \leq b\) for all analytic P-ideals \(\mathcal{I}\). Here \(0 \times \text{FIN}\) is
   \[\{X \subseteq \omega \times \omega : \forall n \in \omega \left[\{m \in \omega : \langle n, m \rangle \in X\} \text{ is finite}\right]\}\]

3. (Fremlin) \(\text{add}^*(\mathcal{Z}_0) = \text{add}(\mathcal{N})\) and \(\text{cof}^*(\mathcal{Z}_0) = \text{cof}(\mathcal{N})\).
Theorem (Hernández-Hernández and Hrušák)

\[ \min\{\text{cov}(\mathcal{N}), b\} \leq \text{cov}^*(\mathcal{Z}_0) \leq \max\{b, \text{non}(\mathcal{N})\} \text{ and } \]
\[ \min\{d, \text{cov}(\mathcal{N})\} \leq \text{non}^*(\mathcal{Z}_0) \leq \max\{d, \text{non}(\mathcal{N})\} \text{ hold.} \]

Question ([2])

Is \( \text{cov}^*(\mathcal{Z}_0) \leq d \)?
This question also has a motivation coming from forcing theory.

**Definition**

Let $\mathcal{V}$ be any ground model and $P \in \mathcal{V}$ be a notion of forcing. Let $I \in \mathcal{V}$ be an ideal on $\omega$. We say that $P$ **diagonalizes** $\mathcal{V} \cap I$ if there exists $\hat{A} \in \mathcal{V}^P$ such that $\Vdash P \hat{A} \in [\omega]^{\omega}$ and for each $X \in \mathcal{V} \cap I$, $\Vdash |X \cap \hat{A}| < \omega$. 

Theorem (Laflamme [3])

Any $F_{\sigma}$ ideal can be diagonalized by a proper $\omega_\omega$-bounding forcing.

Corollary

There is a model where $\text{cov}^*(I) > d$ for every tall $F_{\sigma}$ ideal $I$. 

Dilip Raghavan
Cardinal invariants of density
This question also has a motivation coming from forcing theory.

**Definition**

Let $\mathcal{V}$ be any ground model and $\mathbb{P} \in \mathcal{V}$ be a notion of forcing. Let $\mathcal{I} \in \mathcal{V}$ be an ideal on $\omega$. We say that $\mathbb{P}$ **diagonalizes** $\mathcal{V} \cap \mathcal{I}$ if there exists $\mathbb{A} \in \mathcal{V}^\mathbb{P}$ such that $\Vdash \mathbb{P}\mathbb{A} \in [\omega]^\omega$ and for each $X \in \mathcal{V} \cap \mathcal{I}$, $\Vdash \mathbb{P}\left|X \cap \mathbb{A}\right| < \omega$.

**Theorem (Laflamme [3])**

Any $F_\sigma$ ideal can be diagonalized by a proper $\omega^\omega$-bounding forcing.

**Corollary**

There is a model where $\text{cov}^*(\mathcal{I}) > \mathfrak{d}$ for every tall $F_\sigma$ ideal $\mathcal{I}$. 
Question

Suppose $\mathcal{I} \in V$ is an $F_{\sigma\delta}$ $P$-ideal. Does there exist a proper $\omega^\omega$-bounding $P \in V$ which diagonalizes $V \cap \mathcal{I}$? Is it consistent that $\text{cov}^*(\mathcal{I}) > \mathfrak{d}$ holds for all tall $F_{\sigma\delta}$ $P$-ideals $\mathcal{I}$?

- If you move one level up to $F_{\sigma\delta\sigma}$ ideals, then this totally fails.
- The ideal $\text{FIN} \times \text{FIN}$ is an $F_{\sigma\delta\sigma}$ ideal and any $P$ that diagonalizes it must add a dominating real.
The Results

Theorem (R. and Shelah [4])
\[ \text{cov}^*(\mathcal{Z}_0) \leq \mathfrak{d}. \]

Corollary
Let \( \mathcal{V} \) be any ground model and let \( E \in \mathcal{V} \) be a dominating family of minimal size. If \( \mathbb{P} \in \mathcal{V} \) diagonalizes \( \mathcal{Z}_0 \cap \mathcal{V} \), then \( E \) is no longer a dominating family in \( \mathcal{V}^{\mathbb{P}} \).
The Results

Theorem (R. and Shelah [4])

\[ \text{cov}^*(\mathcal{Z}_0) \leq d. \]

Corollary

Let \( V \) be any ground model and let \( E \in V \) be a dominating family of minimal size. If \( P \in V \) diagonalizes \( \mathcal{Z}_0 \cap V \), then \( E \) is no longer a dominating family in \( V^P \).

Theorem (R.)

\[ \text{cov}^*(\mathcal{Z}_0) \leq \max\{b, s\}. \]

- The proof dualizes to give \( \text{non}^*(\mathcal{Z}_0) \geq \min\{d, r\} \).
Theorem (R.)

Let $\kappa$ be any cardinal. Suppose there exists a function $c : \kappa \times \omega \times \omega \to 2$ such that for any set $A \in [\omega]^{\omega}$ and any partition $\langle X_n : n \in \omega \rangle$ of $\kappa$ into countably many pieces, there exists $n \in \omega$ such that
\[
\forall \sigma \in 2^n \exists k \in A \exists \alpha \in X_n \forall i < n [\sigma(i) = c(\alpha, k, i)].
\]
Then $\text{cov}^*(\mathcal{Z}_0) \leq \max\{b, \kappa\}$.

Claim

If $\kappa = \max\{b, s\}$, then there exists a function $c : \kappa \times \omega \times \omega \to 2$ as in the Theorem.
Open Questions

Question

Is \( \text{cov}^*(\mathcal{Z}_0) \leq b \)?

It is consistent to have \( \text{cov}^*(\mathcal{Z}_0) > s \).

This is because Suslin c.c.c. posets (and their FS iterations) do not increase \( s \).

\( \mathcal{M}(\mathcal{Z}^*0) \) is Suslin c.c.c.

Question

Does \( \text{add}^*(I) = \text{add}(N) \) for all tall analytic P-ideals?
Open Questions

Question

Is $\text{cov}^*(\mathcal{Z}_0) \leq b$?

- It is consistent to have $\text{cov}^*(\mathcal{Z}_0) > s$.
- This is because Suslin c.c.c. posets (and their FS iterations) do not increase $s$.
- $\mathcal{M}(\mathcal{Z}^*_0)$ is Suslin c.c.c.
Open Questions

Question

Is $\text{cov}^*(\mathcal{Z}_0) \leq b$?

- It is consistent to have $\text{cov}^*(\mathcal{Z}_0) > s$.
- This is because Suslin c.c.c. posets (and their FS iterations) do not increase $s$.
- $\mathcal{M}(\mathcal{Z}_0^*)$ is Suslin c.c.c.

Question

Does $\text{add}^*(I) = \text{add}(N)$ for all tall analytic P-ideals?
Bibliography


