Cardinal invariants of density

Dilip Raghavan

National University of Singapore

Computability Theory and Foundations of Mathematics 2015,
Tokyo Institute of Technology, Tokyo, Japan
September 8, 2015

Outline

Basic definitions

Definition

$\mathcal{I} \subseteq \mathcal{P}(\omega)$ *is an* ***ideal on*** ω *if*

1. $\mathcal{I}$ *is closed under subsets and finite unions.*
2. *Every finite subset of* ω *belongs to* $\mathcal{I}$*.*
3. $\omega \notin \mathcal{I}$.

- In this talk I am primarily interested in $\mathcal{I}$ that are definable

- The P-ideals form a special class.

Definition

An ideal $\mathcal{I}$ on ω is called a ***P-ideal*** *if $\mathcal{I}$ is countably directed mod finite. In other words, if $\{a_n : n \in \omega\} \subseteq \mathcal{I}$, then there exists $a \in \mathcal{I}$ such that $\forall n \in \omega\, [a_n \subseteq^* a]$.*

- Here $a \subseteq^* b$ means $a \setminus b$ is finite.

- The P-ideals form a special class.

Definition

*An ideal $\mathcal{I}$ on ω is called a **P-ideal** if $\mathcal{I}$ is countably directed mod finite. In other words, if $\{a_n : n \in \omega\} \subseteq \mathcal{I}$, then there exists $a \in \mathcal{I}$ such that $\forall n \in \omega\,[a_n \subseteq^* a]$.*

- Here $a \subseteq^* b$ means $a \setminus b$ is finite.
- Being a P-ideal has a strong influence on the structure of an ideal $\mathcal{I}$.
- It also influences the possible definable complexity of $\mathcal{I}$.

- $\mathcal{P}(\omega)$ is a Polish space with the usual Cantor topology.
- Sets of the form $\{X \subseteq \omega : n \in X\}$ and $\{X \subseteq \omega : n \notin X\}$ form a sub-basis.
- We can talk about the complexity of $\mathcal{I}$ in the descriptive sense.
- The simplest are the $\mathcal{F}_\sigma$ ideals.

- These have a characterization in terms of sub-measures:

Definition

A function $\phi : \mathcal{P}(\omega) \to [0, \infty]$ is called a ***sub-measure*** *if*

1. $\phi(0) = 0$ *and* $\phi(\{n\}) < \infty$*, for every* $n \in \omega$*;*
2. $X \subseteq Y \implies \phi(X) \leq \phi(Y)$*;*
3. $\phi(X \cup Y) \leq \phi(X) + \phi(Y)$*;*

Definition

A sub-measure ϕ is ***lower semi-continuous (lsc)*** *if for any $X \subseteq \omega$,* $\phi(X) = \lim_{n\to\infty}\phi(X \cap n)$.

Fact (Mazur)

An ideal $\mathcal{I}$ on ω is F_σ iff $\mathcal{I} = \mathrm{Fin}(\phi) = \{X \subseteq \omega : \phi(X) < \infty\}$.

Example

*$\mathcal{I}_{\frac{1}{n}}$ is the ideal of **summable sets**. That is*

$$\mathcal{I}_{\frac{1}{n}} = \left\{X \subseteq \omega : \sum_{n \in X} \frac{1}{n} < \infty\right\}$$

- $\mathcal{I}_{\frac{1}{n}}$ is actually a P-ideal.
- The sub-measure here is just $\phi(X) = \sum_{n \in X} \frac{1}{n}$.
- Can replace $\frac{1}{n}$ by any divergent series (the ideals are quite different though!).

Example

$\mathcal{ED}$ is the ideal on $\omega \times \omega$ generated by the vertical columns and graphs of functions. That is $\mathcal{ED}$ =

$$\{X \subseteq \omega \times \omega : \exists k, l \in \omega \forall n > k \, [|\{m \in \omega : \langle n, m \rangle \in X\}| \leq l]\}$$

- This is an F_σ ideal which is not P.

- Moving up the complexity hierarchy, it turns out that every analytic P-ideal is $F_{\sigma\delta}$.
- So at least for P-ideals, there is nothing between $F_{\sigma\delta}$ and Π^1_1.

- Moving up the complexity hierarchy, it turns out that every analytic P-ideal is $F_{\sigma\delta}$.
- So at least for P-ideals, there is nothing between $F_{\sigma\delta}$ and Π^1_1.

Theorem (Solecki)

Let $\mathcal{I}$ be an ideal on ω.

1. *$\mathcal{I}$ is an analytic P-ideal iff there exists a lower semi-continuous sub-measure ϕ such that $\mathcal{I} = \mathrm{Exh}(\phi) = \{X \subseteq \omega : \lim_{n\to\infty}\phi(X \setminus n) = 0\}$.*
2. *$\mathcal{I}$ is an $\mathcal{F}_\sigma$ P-ideal iff there exists a lower semi-continuous sub-measure ϕ such that $\mathcal{I} = \mathrm{Fin}(\phi) = \mathrm{Exh}(\phi)$.*

$\mathrm{Exh}(\phi)$ *is always an $F_{\sigma\delta}$ P-ideal.*

Example

*A set $A \subseteq \omega$ is said to have **asymptotic density** 0 if $\lim_{n \to \infty} \frac{|A \cap n|}{n} = 0$.*

$$\mathcal{Z}_0 = \left\{ A \subseteq \omega : \lim_{n \to \infty} \frac{|A \cap n|}{n} = 0 \right\}.$$

- This an $F_{\sigma\delta}$ P-ideal.
- Suppose $\{a_n : n \in \omega\} \subseteq \mathcal{Z}_0$.
- WLOG they are pairwise disjoint.

- Let $b_n = \bigcup_{m \leq n} a_m$ and let k_n be minimal such that for all $k \geq k_n$, $\frac{|b_n \cap k|}{k} \leq 2^{-n}$.
- Let $a = \bigcup_{n \in \omega} (a_n \setminus k_n)$.
- This set a works.

Three basic invariants

- Cardinals invariants are cardinal between $\aleph_1$ and $\mathfrak{c} = 2^{\aleph_0}$.
- They identify places where basic diagonalization arguments first fail.

Three basic invariants

- Cardinals invariants are cardinal between $\aleph_1$ and $\mathfrak{c} = 2^{\aleph_0}$.
- They identify places where basic diagonalization arguments first fail.

Definition

*For $f, g \in \omega^\omega$, $f <^* g$ means that $|\{n \in \omega : g(n) \leq f(n)\}| < \omega$. A set $F \subseteq \omega^\omega$ is said to be **unbounded** if there does not exist $g \in \omega^\omega$ such that $\forall f \in F\,[f <^* g]$. A set $F \subseteq \omega^\omega$ is said to be **dominating or cofinal** if $\forall f \in \omega^\omega \exists g \in F\,[f <^* g]$.*

Three basic invariants

- Cardinals invariants are cardinal between $\aleph_1$ and $\mathfrak{c} = 2^{\aleph_0}$.
- They identify places where basic diagonalization arguments first fail.

Definition

*For $f, g \in \omega^\omega$, $f <^* g$ means that $|\{n \in \omega : g(n) \leq f(n)\}| < \omega$. A set $F \subseteq \omega^\omega$ is said to be **unbounded** if there does not exist $g \in \omega^\omega$ such that $\forall f \in F\, [f <^* g]$. A set $F \subseteq \omega^\omega$ is said to be **dominating or cofinal** if $\forall f \in \omega^\omega \exists g \in F\, [f <^* g]$.*

Definition

*For $a, b \in \mathcal{P}(\omega)$ we say that a **splits** b if both $b \cap a$ and $b \cap (\omega \setminus a)$ are infinite. A family $F \subseteq \mathcal{P}(\omega)$ is called a **splitting family** if $\forall b \in [\omega]^\omega \exists a \in F\, [a \text{ splits } b]$.*

Definition

We define the cardinal invariants $\mathfrak{b}$, $\mathfrak{d}$, *and* $\mathfrak{s}$ *as follows:*

$$\mathfrak{b} = \min\{|F| : F \subseteq \omega^\omega \wedge F \textit{ is unbounded}\};$$
$$\mathfrak{d} = \min\{|F| : F \subseteq \omega^\omega \wedge F \textit{ is dominating}\};$$
$$\mathfrak{s} = \min\{|F| : F \subseteq \mathcal{P}(\omega) \wedge F \textit{ is a splitting family}\}.$$

Fact

$\aleph_1 \leq \max\{\mathfrak{b}, \mathfrak{s}\} \leq \mathfrak{d} \leq \mathfrak{c}$.

- This is all that can be proved in ZFC.

- We consider cardinal invariants associated with analytic P-ideals.
- Two possibilities: invariants associated with the quotient $\mathcal{P}(\omega)/\mathcal{I}$ and cardinals associated with $\mathcal{I}$ itself.
- Former is similar to $\mathcal{P}(\omega)/\mathrm{FIN}$.
- The latter involves possibilities that don't make sense for FIN because FIN is not a tall ideal.

Definition

*Recall that an ideal $\mathcal{I}$ on ω is **tall** if it has the property that $\forall a \in [\omega]^{\omega} \exists b \in [a]^{\omega} \, [b \in \mathcal{I}]$.*

- When $\mathcal{I}$ is a tall P-ideal, we can define invariants associated with $\mathcal{I}$ that don't make sense for FIN.
- There are many interesting open problems about invariants associated with $\mathcal{P}(\omega)/\mathcal{I}$ (not our topic for today, but . . .).

- When $\mathcal{I}$ is a tall P-ideal, we can define invariants associated with $\mathcal{I}$ that don't make sense for FIN.
- There are many interesting open problems about invariants associated with $\mathcal{P}(\omega)/\mathcal{I}$ (not our topic for today, but ...).

Definition

A family $F \subseteq \mathcal{P}(\omega)$ is ***splitting for*** $\mathcal{P}(\omega)/\mathcal{I}_{\frac{1}{n}}$ *if*

$$\forall b \in \mathcal{I}_{\frac{1}{n}}^{+} \exists a \in F \left[b \cap a \in \mathcal{I}_{\frac{1}{n}}^{+} \wedge b \cap (\omega \setminus a) \in \mathcal{I}_{\frac{1}{n}}^{+} \right].$$

- When $\mathcal{I}$ is a tall P-ideal, we can define invariants associated with $\mathcal{I}$ that don't make sense for FIN.
- There are many interesting open problems about invariants associated with $\mathcal{P}(\omega)/\mathcal{I}$ (not our topic for today, but ...).

Definition

*A family $F \subseteq \mathcal{P}(\omega)$ is **splitting for** $\mathcal{P}(\omega)/\mathcal{I}_{\frac{1}{n}}$ if*

$$\forall b \in \mathcal{I}^{+}_{\frac{1}{n}} \exists a \in F \left[b \cap a \in \mathcal{I}^{+}_{\frac{1}{n}} \wedge b \cap (\omega \setminus a) \in \mathcal{I}^{+}_{\frac{1}{n}} \right].$$

Definition

The analogue of $\mathfrak{s}$ for $\mathcal{P}(\omega)/\mathcal{I}_{\frac{1}{n}}$ is:

$$\mathfrak{s}_{\frac{1}{n}} = \min \left\{ |F| : F \subseteq \mathcal{P}(\omega) \text{ is splitting for } \mathcal{P}(\omega)/\mathcal{I}_{\frac{1}{n}} \right\}.$$

Theorem (Brendle)

It is consistent to have $\mathfrak{s}_{\frac{1}{n}} < \mathfrak{s}$.

Question

Is $\mathfrak{s} < \mathfrak{s}_{\frac{1}{n}}$ *consistent?*

Question

Is $\mathfrak{h} < \mathfrak{h}_{\frac{1}{n}}$ *consistent?*

Definition

When $\mathcal{I}$ is a tall P-ideal on ω you can define the following:

$$\mathrm{add}^*(\mathcal{I}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \wedge \forall b \in \mathcal{I} \exists a \in \mathcal{F} \, [a \not\subseteq^* b]\},$$
$$\mathrm{cov}^*(\mathcal{I}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \wedge \forall a \in [\omega]^\omega \exists b \in \mathcal{F} \, [|a \cap b| = \omega]\},$$
$$\mathrm{cof}^*(\mathcal{I}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \wedge \forall b \in \mathcal{I} \exists a \in \mathcal{F} \, [b \subseteq^* a]\},$$
$$\mathrm{non}^*(\mathcal{I}) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^\omega \wedge \forall b \in \mathcal{I} \exists a \in \mathcal{F} \, [|a \cap b| < \omega]\}.$$

Definition

When $\mathcal{I}$ is a tall P-ideal on ω you can define the following:

$$\begin{aligned}
\mathrm{add}^*(\mathcal{I}) &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \wedge \forall b \in \mathcal{I} \exists a \in \mathcal{F}\, [a \not\subseteq^* b]\},\\
\mathrm{cov}^*(\mathcal{I}) &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \wedge \forall a \in [\omega]^\omega \exists b \in \mathcal{F}\, [|a \cap b| = \omega]\},\\
\mathrm{cof}^*(\mathcal{I}) &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \wedge \forall b \in \mathcal{I} \exists a \in \mathcal{F}\, [b \subseteq^* a]\},\\
\mathrm{non}^*(\mathcal{I}) &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq [\omega]^\omega \wedge \forall b \in \mathcal{I} \exists a \in \mathcal{F}\, [|a \cap b| < \omega]\}.
\end{aligned}$$

- If $\mathcal{I}$ were not a P-ideal, $\mathrm{add}^*(\mathcal{I})$ would be ω.
- If $\mathcal{I}$ were not tall, then $\mathrm{cov}^*(\mathcal{I})$ would be undefined, and $\mathrm{non}^*(\mathcal{I})$ would be 1.

- These invariants were investigated by Hernández-Hernández and Hrušák [2] and also by Brendle and Shelah [1].
- Terminology is based on analogy with the following definitions which make sense for any ideal whatsoever.

Definition

Let $\mathcal{I}$ be any ideal on a set X. Define

$$\operatorname{add}(\mathcal{I}) = \min\left\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \wedge \bigcup \mathcal{F} \notin \mathcal{I}\right\},$$
$$\operatorname{cov}(\mathcal{I}) = \min\left\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \wedge \bigcup \mathcal{F} = X\right\},$$
$$\operatorname{cof}(\mathcal{I}) = \min\left\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \wedge \forall B \in \mathcal{I} \exists A \in \mathcal{F}\, [B \subseteq A]\right\},$$
$$\operatorname{non}(\mathcal{I}) = \{|Y| : Y \subseteq X \wedge Y \notin \mathcal{I}\}.$$

- These invariants were investigated by Hernández-Hernández and Hrušák [2] and also by Brendle and Shelah [1].
- Terminology is based on analogy with the following definitions which make sense for any ideal whatsoever.

Definition

Let $\mathcal{I}$ be any ideal on a set X. Define

$$\begin{aligned}
\mathrm{add}(\mathcal{I}) &= \min\left\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \wedge \bigcup \mathcal{F} \notin \mathcal{I}\right\},\\
\mathrm{cov}(\mathcal{I}) &= \min\left\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \wedge \bigcup \mathcal{F} = X\right\},\\
\mathrm{cof}(\mathcal{I}) &= \min\left\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \wedge \forall B \in \mathcal{I} \exists A \in \mathcal{F}\, [B \subseteq A]\right\},\\
\mathrm{non}(\mathcal{I}) &= \left\{|Y| : Y \subseteq X \wedge Y \notin \mathcal{I}\right\}.
\end{aligned}$$

- $\mathrm{add}(\mathcal{I})$ and $\mathrm{cof}(\mathcal{I})$ are duals. So are $\mathrm{cov}(\mathcal{I})$ and $\mathrm{non}(\mathcal{I})$.

- For each $a \in \mathcal{P}(\omega)$, let $\hat{a} = \{b \subseteq \omega : |a \cap b| = \omega\}$.
- For each $a \in \mathcal{P}(\omega)$, let $\hat{a} = \{b \subseteq \omega : |a \cap b| = \omega\}$.
- For a tall ideal $\mathcal{I}$, $\hat{\mathcal{I}} = \{X \subseteq \mathcal{P}(\omega) : \exists a \in \mathcal{I}\,[X \subseteq \hat{a}]\}$ is an ideal on $\mathcal{P}(\omega)$ generated by Borel sets.

- For each $a \in \mathcal{P}(\omega)$, let $\hat{a} = \{b \subseteq \omega : |a \cap b| = \omega\}$.
- For each $a \in \mathcal{P}(\omega)$, let $\hat{a} = \{b \subseteq \omega : |a \cap b| = \omega\}$.
- For a tall ideal $\mathcal{I}$, $\hat{\mathcal{I}} = \{X \subseteq \mathcal{P}(\omega) : \exists a \in \mathcal{I}\, [X \subseteq \hat{a}]\}$ is an ideal on $\mathcal{P}(\omega)$ generated by Borel sets.
- $\mathcal{I}$ is a P-ideal iff $\hat{\mathcal{I}}$ is a σ-ideal.
- $\mathrm{add}(\hat{\mathcal{I}}) = \mathrm{add}^*(\mathcal{I})$, $\mathrm{cov}(\hat{\mathcal{I}}) = \mathrm{cov}^*(\mathcal{I})$, $\mathrm{cof}(\hat{\mathcal{I}}) = \mathrm{cof}^*(\mathcal{I})$, $\mathrm{non}(\hat{\mathcal{I}}) = \mathrm{non}^*(\mathcal{I})$ hold.

- The Tukey and the Katětov orderings are relevant to these invariants.

Definition

Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω. Recall that $\mathcal{I}$ is ***Katětov below*** *$\mathcal{J}$ or $\mathcal{I} \leq_K \mathcal{J}$ if there is a function $f : \omega \to \omega$ such that $\forall a \in \mathcal{I}\left[f^{-1}(a) \in \mathcal{J}\right]$.*

Definition

We say that $\langle \mathcal{I}, \subseteq^* \rangle$ is ***Tukey below*** *$\langle \mathcal{J}, \subseteq^* \rangle$ and we write $\mathcal{I} \leq_T^* \mathcal{J}$ if there is a map $\varphi : \mathcal{I} \to \mathcal{J}$ such that if $X \subseteq \mathcal{I}$ any set that does not have an upper bound in the partial order $\langle \mathcal{I}, \subseteq^* \rangle$, then $\varphi'' X$ does not have an upper bound in the partial order $\langle \mathcal{J}, \subseteq^* \rangle$.*

- The Tukey and the Katětov orderings are relevant to these invariants.

Definition

Let $\mathcal{I}$ and $\mathcal{J}$ be ideals on ω. Recall that $\mathcal{I}$ is ***Katětov below*** *$\mathcal{J}$ or $\mathcal{I} \leq_K \mathcal{J}$ if there is a function $f : \omega \to \omega$ such that $\forall a \in \mathcal{I} \left[f^{-1}(a) \in \mathcal{J} \right]$.*

Definition

We say that $\langle \mathcal{I}, \subseteq^* \rangle$ is ***Tukey below*** *$\langle \mathcal{J}, \subseteq^* \rangle$ and we write $\mathcal{I} \leq_T^* \mathcal{J}$ if there is a map $\varphi : \mathcal{I} \to \mathcal{J}$ such that if $X \subseteq \mathcal{I}$ any set that does not have an upper bound in the partial order $\langle \mathcal{I}, \subseteq^* \rangle$, then $\varphi'' X$ does not have an upper bound in the partial order $\langle \mathcal{J}, \subseteq^* \rangle$.*

- $\mathcal{I} \leq_K \mathcal{J}$ implies both that $\mathrm{cov}^*(\mathcal{I}) \geq \mathrm{cov}^*(\mathcal{J})$ and that $\mathrm{non}^*(\mathcal{I}) \leq \mathrm{non}^*(\mathcal{J})$.
- If $\mathcal{I} \leq_T^* \mathcal{J}$, then $\mathrm{add}^*(\mathcal{I}) \geq \mathrm{add}^*(\mathcal{J})$ and $\mathrm{cof}^*(\mathcal{I}) \leq \mathrm{cof}^*(\mathcal{J})$.

- Summary of some known results:

Fact

Let $\mathcal{I}$ be a tall P-ideal on ω.

1. $\aleph_1 \leq \text{add}^*(\mathcal{I}) \leq \min\{\text{non}^*(\mathcal{I}), \text{cov}^*(\mathcal{I})\} \leq \max\{\text{non}^*(\mathcal{I}), \text{cov}^*(\mathcal{I})\} \leq \text{cof}^*(\mathcal{I}) \leq \mathfrak{c}$.
2. $\mathfrak{p} \leq \text{cov}^*(\mathcal{I})$.

Theorem

The following hold:

1. $\mathrm{add}^*\left(\mathcal{I}_{\frac{1}{n}}\right) = \mathrm{add}(\mathcal{N})$.
2. *(Todorcevic) For every analytic P-ideal* $\mathcal{I}$, $0 \times \mathrm{FIN} \leq_T^* \mathcal{I} \leq_T^* \mathcal{I}_{\frac{1}{n}}$. *Therefore* $\mathrm{add}(\mathcal{N}) \leq \mathrm{add}^*(\mathcal{I}) \leq \mathfrak{b}$ *for all analytic P-ideals* $\mathcal{I}$. *Here* $0 \times \mathrm{FIN}$ *is*
$$\{X \subseteq \omega \times \omega : \forall n \in \omega\, [\{m \in \omega : \langle n, m\rangle \in X\} \text{ is finite}]\}$$
3. *(Fremlin)* $\mathrm{add}^*(\mathcal{Z}_0) = \mathrm{add}(\mathcal{N})$ *and* $\mathrm{cof}^*(\mathcal{Z}_0) = \mathrm{cof}(\mathcal{N})$.

Theorem (Hernández-Hernández and Hrušák)

$\min\{\operatorname{cov}(\mathcal{N}), \mathfrak{b}\} \leq \operatorname{cov}^*(\mathcal{Z}_0) \leq \max\{\mathfrak{b}, \operatorname{non}(\mathcal{N})\}$ *and*
$\min\{\mathfrak{d}, \operatorname{cov}(\mathcal{N})\} \leq \operatorname{non}^*(\mathcal{Z}_0) \leq \max\{\mathfrak{d}, \operatorname{non}(\mathcal{N})\}$ *hold.*

Question ([2])

Is $\operatorname{cov}^*(\mathcal{Z}_0) \leq \mathfrak{d}$*?*

- This question also has a motivation coming from forcing theory.

Definition

Let $\mathbf{V}$ be any ground model and $\mathbb{P} \in \mathbf{V}$ be a notion of forcing. Let $\mathcal{I} \in \mathbf{V}$ be an ideal on ω. We say that $\mathbb{P}$ ***diagonalizes*** *$\mathbf{V} \cap \mathcal{I}$ if there exists $\mathring{A} \in \mathbf{V}^{\mathbb{P}}$ such that $\Vdash_{\mathbb{P}} \mathring{A} \in [\omega]^{\omega}$ and for each $X \in \mathbf{V} \cap \mathcal{I}$, $\Vdash_{\mathbb{P}} \left|X \cap \mathring{A}\right| < \omega$.*

- This question also has a motivation coming from forcing theory.

Definition

*Let $\mathbf{V}$ be any ground model and $\mathbb{P} \in \mathbf{V}$ be a notion of forcing. Let $\mathcal{I} \in \mathbf{V}$ be an ideal on ω. We say that $\mathbb{P}$ **diagonalizes** $\mathbf{V} \cap \mathcal{I}$ if there exists $\mathring{A} \in \mathbf{V}^{\mathbb{P}}$ such that $\Vdash_{\mathbb{P}} \mathring{A} \in [\omega]^{\omega}$ and for each $X \in \mathbf{V} \cap \mathcal{I}$, $\Vdash_{\mathbb{P}} \left|X \cap \mathring{A}\right| < \omega$.*

Theorem (Laflamme [3])

Any F_σ ideal can be diagonalized by a proper ω^ω-bounding forcing.

Corollary

There is a model where $\mathrm{cov}^*(\mathcal{I}) > \mathfrak{d}$ for every tall F_σ ideal $\mathcal{I}$.

Question

Suppose $\mathcal{I} \in \mathbf{V}$ is an $F_{\sigma\delta}$ P-ideal. Does there exist a proper ω^{ω}-bounding $\mathbb{P} \in \mathbf{V}$ which diagonalizes $\mathbf{V} \cap \mathcal{I}$? Is it consistent that $\mathrm{cov}^{*}(\mathcal{I}) > \mathfrak{d}$ holds for all tall $F_{\sigma\delta}$ P-ideals $\mathcal{I}$?

- If you move one level up to $F_{\sigma\delta\sigma}$ ideals, then this totally fails.
- The ideal $\mathrm{FIN} \times \mathrm{FIN}$ is an $F_{\sigma\delta\sigma}$ ideal and any $\mathbb{P}$ that diagonalizes it must add a dominating real.

The Results

Theorem (R. and Shelah [4])

$\mathrm{cov}^*(\mathcal{Z}_0) \leq \mathfrak{d}$.

Corollary

Let $\mathbf{V}$ be any ground model and let $E \in \mathbf{V}$ be a dominating family of minimal size. If $\mathbb{P} \in \mathbf{V}$ diagonalizes $\mathcal{Z}_0 \cap \mathbf{V}$, then E is no longer a dominating family in $\mathbf{V}^{\mathbb{P}}$.

The Results

Theorem (R. and Shelah [4])

$\mathrm{cov}^*(\mathcal{Z}_0) \leq \mathfrak{d}$.

Corollary

Let $\mathbf{V}$ be any ground model and let $E \in \mathbf{V}$ be a dominating family of minimal size. If $\mathbb{P} \in \mathbf{V}$ diagonalizes $\mathcal{Z}_0 \cap \mathbf{V}$, then E is no longer a dominating family in $\mathbf{V}^{\mathbb{P}}$.

Theorem (R.)

$\mathrm{cov}^*(\mathcal{Z}_0) \leq \max\{\mathfrak{b}, \mathfrak{s}\}$.

- The proof dualizes to give $\mathrm{non}^*(\mathcal{Z}_0) \geq \min\{\mathfrak{d}, \mathfrak{r}\}$.

Theorem (R.)

Let κ be any cardinal. Suppose there exists a function $c : \kappa \times \omega \times \omega \to 2$ such that for any set $A \in [\omega]^{\omega}$ and any partition $\langle X_n : n \in \omega \rangle$ of κ into countably many pieces, there exists $n \in \omega$ such that $\forall \sigma \in 2^n \exists k \in A \exists \alpha \in X_n \forall i < n \, [\sigma(i) = c(\alpha, k, i)]$. Then $\mathrm{cov}^*(\mathcal{Z}_0) \leq \max\{\mathfrak{b}, \kappa\}$.

Claim

If $\kappa = \max\{\mathfrak{b}, \mathfrak{s}\}$, then there exists a function $c : \kappa \times \omega \times \omega \to 2$ as in the Theorem.

Open Questions

Question

Is $\mathrm{cov}^*(\mathcal{Z}_0) \leq \mathfrak{b}$*?*

Open Questions

Question

Is $\mathrm{cov}^*(\mathcal{Z}_0) \leq \mathfrak{b}$*?*

- It is consistent to have $\mathrm{cov}^*(\mathcal{Z}_0) > \mathfrak{s}$.
- This is because Suslin c.c.c. posets (and their FS iterations) do not increase $\mathfrak{s}$.
- $\mathbb{M}(\mathcal{Z}_0^*)$ is Suslin c.c.c.

Open Questions

Question

Is $\mathrm{cov}^*(\mathcal{Z}_0) \leq \mathfrak{b}$*?*

- It is consistent to have $\mathrm{cov}^*(\mathcal{Z}_0) > \mathfrak{s}$.
- This is because Suslin c.c.c. posets (and their FS iterations) do not increase $\mathfrak{s}$.
- $\mathbb{M}(\mathcal{Z}_0^*)$ is Suslin c.c.c.

Question

Does $\mathrm{add}^*(\mathcal{I}) = \mathrm{add}(\mathcal{N})$ *for all tall analytic P-ideals?*

Bibliography

- J. Brendle and S. Shelah, *Ultrafilters on ω—their ideals and their cardinal characteristics*, Trans. Amer. Math. Soc. **351** (1999), no. 7, 2643–2674.
- F. Hernández-Hernández and M. Hrušák, *Cardinal invariants of analytic P-ideals*, Canad. J. Math. **59** (2007), no. 3, 575–595.
- C. Laflamme, *Zapping small filters*, Proc. Amer. Math. Soc. **114** (1992), no. 2, 535–544.
- D. Raghavan and S. Shelah, *Two inequalities between cardinal invariants*, Preprint (2015).