Coloring the rationals in reverse mathematics

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(joint work with Ludovic Patey)

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Outline

Introduction

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Beyond the big five

Big five and the Zoo. Ramsey's theorem for pairs RT^2_2 is the first example of statement not equivalent to one of the main systems of reverse mathematics. Many consequences of RT^2_2 have been studied, leading to many independent statements.

However, there are no natural statements between RT^2_2 and ACA_0. The only known candidate is the tree theorem for pairs TT^2_2.

We discuss another candidate, arguably more natural. This is a partition theorem due to Erdős and Rado, and it's a strengthening of Ramsey's theorem for pairs.

Theorem (Ramsey's Theorem for pairs and two colors)

RT^2_2 *Every coloring* $f : [\mathbb{N}]^2 \to 2$ *has an* **infinite homogeneous** *set.*

Theorem (Pigeonhole Principle on natural numbers)

$\mathsf{RT}^1_{<\infty}$ *Let* $k \in \mathbb{N}$. *Every coloring* $f : \mathbb{N} \to k$ *has an* **infinite homogeneous** *set.*

Theorem (Erdős-Rado Theorem)

$(\aleph_0, \eta)^2$ *Every coloring $f : [\mathbb{Q}]^2 \to 2$ has either an* **infinite** 0-**homogeneous** *set or a* **dense** 1-**homogeneous** *set.*

Theorem (Pigeonhole principle on rationals)

$(\eta)^1_{<\infty}$ *Let $k \in \mathbb{N}$. Every coloring $f : \mathbb{Q} \to k$ has a* **dense homogeneous** *set.*

Theorem (Tree Theorem for pairs and two colors)

TT^2_2 *Every coloring* $f : [2^{<\mathbb{N}}]^2 \to 2$ *has a* **homogeneous tree**.

Theorem (Pigeonhole Principle on trees)

TT^1 *Let* $k \in \mathbb{N}$. *Every coloring* $f : 2^{<\mathbb{N}} \to k$ *has a* **homogeneous tree**.

Lemma (RCA_0)

- $\mathsf{ACA}_0 \rightarrow (\aleph_0, \eta)^2 \rightarrow \mathsf{RT}^2_2$
- $(\aleph_0, \eta)^2 \rightarrow (\eta)^1_{<\infty}$
- $\mathsf{I}\Sigma^0_2 \rightarrow (\eta)^1_{<\infty} \rightarrow \mathsf{B}\mathbf{\Sigma}^0_2$

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Theorem (F. and Patey)

- $\mathsf{RCA}_0 + \mathsf{B}\boldsymbol{\Sigma}^0_2 \nvdash (\eta)^1_{<\infty}$
- $(\aleph_0, \eta)^2 \not\leq_c \mathsf{RT}^2_{<\infty}$

We separate $(\eta)^1_{<\infty}$ from $\mathsf{B}\mathbf{\Sigma}^0_2$ by adapting the model-theoretic proof of Corduan, Groszek, and Mileti that separates TT^1 from $\mathsf{B}\mathbf{\Sigma}^0_2$.

We separate $(\eta)^1_{<\infty}$ from $\mathsf{B}\boldsymbol{\Sigma}^0_2$ by adapting the model-theoretic proof of Corduan, Groszek, and Mileti that separates TT^1 from $\mathsf{B}\boldsymbol{\Sigma}^0_2$.

Basically, in a model of $\mathsf{RCA}_0 + \neg\,\mathsf{I}\Sigma^0_2$, there is a real X and an X-recursive instance of $(\eta)^1_{<\infty}$ with no X-recursive solutions.

The proof consists of two steps.

Lemma (Step 1)

In a model M of RCA_0, for every $X \in M$, there is a uniform X-recursive way, given finitely many X-r.e. subsets of $\mathbb{Q}$, to compute a 2-coloring $f : \mathbb{Q} \to 2$ so as to defeat all the given potential homogeneous sets.

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To obtain such a result, we use a combinatorial feature of $(\eta)^1_{<\infty}$ shared by TT^1.

The basic idea is as follows. We are given many dense potential sets W_e^X with $e < n$, and we build f by stages.

The basic strategy to diagonalize against a single W_e^X is to wait until we see 2 disjoint intervals with end-points in W_e^X and then color the two intervals with 0 and 1 respectively. This works in isolation.

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We take care of all W_e^X's by fixing $4n$ disjoint intervals with end-points in W_e^X for every W_e^X that outputs $4n+1$ points (we say that W_e^X requires attention). By a simple combinatorial argument, from $k \leq n$ tuples of $4n$ disjoint intervals we can select a pair from each tuple so as to have $2k$ disjoint intervals.

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At any stage we color every current pair of intervals with 0 and 1 respectively. Since there are finitely many W_e^X's, we eventually stabilize on some pair for each W_e^X that requires attention.

Lemma (Step 2)

Let M be a model of RCA_0 and suppose that M does not satisfy $\mathsf{I}\Sigma^0_2(X)$ for some $X \subseteq M$. Then there is an X-recursive coloring f of $\mathbb{Q}$ into finitely many colors such that no X-recursive dense set is homogeneous for f.

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The failure of $\mathsf{I}\Sigma^0_2(X)$ implies that there is an X-recursive function $h\colon \mathbb{N}^2 \to \mathbb{N}$ such that for some number a, the range of the partial function $h(y) = \lim_{s\to\infty} h(y,s)$ is unbounded on $\{y : y < a\}$.

Theorem

Let P be a Π^1_1 sentence. Then $\mathsf{RCA}_0 + P \vdash (\eta)^1_{<\infty}$ *if and only if* $\mathsf{RCA}_0 + P \vdash \mathsf{I}\Sigma^0_2$*. In particular,* $\mathsf{RCA}_0 + \mathsf{B}\boldsymbol{\Sigma}^0_2 \nvdash (\eta)^1_{<\infty}$.

Theorem

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Proof sketch.

Let M be a model of $\mathsf{RCA}_0 + P$ where $\mathsf{I}\Sigma^0_2$ fails, and $X \in M$ as above. Then $\boldsymbol{\Delta}^0_2(X)$ is a model of $\mathsf{RCA}_0 + P$ where $(\eta)^1_{<\infty}$ fails. □

Most implications of the form $Q \to P$ over RCA_0, where P and Q are $\mathbf{\Pi}^1_2$ statements, make use only of one Q-instance to solve a P-instance. This is the notion of computable reducibility.

Definition

Fix two Π^1_2 statements P and Q. P is **computably reducible** to Q (written $P \leq_c Q$) if every P-instance I computes a Q-instance J such that, for every solution S to J, $I \oplus S$ computes a solution to I.

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To show that $P \not\leq_c Q$, it is “enough” to produce a computable P-instance I such that every computable Q-instance has a solution that does not compute a solution to I.

$P \leq_c Q$ does not mean that $\mathsf{RCA}_0 \vdash Q \to P$. In some cases, it is possible to obtain a separation over ω-models from a one-step non-reduction.

- ADS does not imply CAC over RCA_0 (Lerman, Solomon, and Towsner)
- EM does not imply RT^2_2 over RCA_0 (Lerman, Solomon, and Towsner)
- RT^2_2 does not imply TT^2_2 over RCA_0 (Patey)

The above results use a general framework.

We prove that $(\aleph_0, \eta)^2 \not\leq_c \mathsf{RT}^2_{<\infty}$. However, we are not able to generalize this result to a separation over ω-models.

Why?

Basically, we want to produce an instance $f\colon [\mathbb{Q}]^2 \to 2$ of $(\aleph_0, \eta)^2$ and solve instances of RT^2_2 without computing solutions to f. We can view this as a game.

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Given an instance g of RT^2_2 we are trying to build a solution H to g which does not compute a solution to f. We regard f as our opponent. So, suppose we want to diagonalize against $\Phi_0^{g \oplus H}$ and $\Phi_1^{g \oplus H}$, where $\Phi_i^{g \oplus H}$ is a potential homogeneous set of color i. Our opponent f commits to make $\Phi_0^{g \oplus H}$ infinite or $\Phi_1^{g \oplus H}$ dense.

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In the case of TT^2_2, our opponent commits to build a full binary tree in either case.

This half commitment property is the main combinatorial difference between the two principles that prevents us from adapting the proof for TT^2_2.

To show that $(\aleph_0, \eta)^2$ does not computably reduce to $RT^2_{<\infty}$, we consider the asymmetric version of $(\eta)^1_{<\infty}$.

$(\aleph_0, \eta)^1$ For every partition $A_0 \cup A_1 = \mathbb{Q}$ there is either an infinite subset of A_0 or a dense subset of A_1.

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$(\aleph_0, \eta)^1$ For every partition $A_0 \cup A_1 = \mathbb{Q}$ there is either an infinite subset of A_0 or a dense subset of A_1.

Theorem (F. and Patey)

There is a Δ^0_2 instance $A_0 \cup A_1 = \mathbb{Q}$ of $(\aleph_0, \eta)^1$ such that every computable coloring $g\colon [\omega]^2 \to k$ has an infinite homogeneous set H that does not compute a solution to $A_0 \cup A_1 = \mathbb{Q}$.

Corollary

There is a computable coloring $f : [\mathbb{Q}]^2 \to 2$ *such that every computable coloring* $g : [\omega]^2 \to k$ *has an infinite homogeneous set* H *that does not compute a solution to* f*.*

Proof.

Let $f(x, s)$ be such that $f(x) = \lim_s f(x, s)$ exists and $x \in A_{f(x)}$. □

The fairness notion

We design a **fairness property** for instances $A_0 \cup A_1 = \mathbb{Q}$ of $(\aleph_0, \eta)^1$.

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More precisely:

(*F*) Given $f\colon [\omega]^2 \to k$, we are able to build infinite homogeneous sets $G_0, \dots, G_{k-1}$, where G_i is homogeneous with color i, such that for all k-tuples of Turing functionals $\Phi_0, \dots, \Phi_{k-1}$, if every $\Phi_i^{G_i}$ is **large**, then one of them is not a solution to $A_0 \cup A_1 = \mathbb{Q}$.

The fairness notion for $(\aleph_0, \eta)^2$ is very technical. In general, it depends on the combinatorics of the problem (see CAC and TT^2_2).

- If an instance $A_0 \cup A_1 = \mathbb{Q}$ of $(\aleph_0, \eta)^1$ is fair with respect to a Scott set $\mathcal{S}$ of reals ((*F*) holds for every $f \in \mathcal{S}$), then every instance $f \in \mathcal{S}$ of $\mathsf{RT}^2_{<\infty}$ has a solution that compute neither an infinite subset of A_0 nor a dense subset of A_1.
- The solutions to instances of $\mathsf{RT}^2_{<\infty}$ are built by using Mathias forcing over Scott sets.
- We can produce a Δ^2_0 instance of $(\aleph_0, \eta)^1$ as above.

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Seetapun's argument does not work for $(\aleph_0, \eta)^2$. Actually, there is no forcing notion to build solutions to any instance of $(\aleph_0, \eta)^2$.

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References

Emanuele Frittaion and Ludovic Patey. *Coloring the rationals in reverse mathematics*. Submitted, 2015. Preprint on arXiv.

Thanks for your attention