Coloring the rationals in reverse mathematics

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(joint work with Ludovic Patey)

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Outline

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Beyond the big five

**Big five and the Zoo.** Ramsey’s theorem for pairs $\text{RT}^2_2$ is the first example of statement not equivalent to one of the main systems of reverse mathematics. Many consequences of $\text{RT}^2_2$ have been studied, leading to many independent statements.

However, there are no natural statements between $\text{RT}^2_2$ and $\text{ACA}_0$. The only known candidate is the tree theorem for pairs $\text{TT}^2_2$.

We discuss another candidate, arguably more natural. This is a partition theorem due to Erdős and Rado, and it’s a strengthening of Ramsey’s theorem for pairs.
Theorem (Ramsey’s Theorem for pairs and two colors)

$$RT^2_2 \quad \text{Every coloring } f : [\mathbb{N}]^2 \rightarrow 2 \text{ has an infinite homogeneous set.}$$

Theorem (Pigeonhole Principle on natural numbers)

$$RT^1_{<\infty} \quad \text{Let } k \in \mathbb{N}. \text{ Every coloring } f : \mathbb{N} \rightarrow k \text{ has an infinite homogeneous set.}$$
Theorem (Erdős-Rado Theorem)

\((\aleph_0, \eta)^2\) Every coloring \(f : [\mathbb{Q}]^2 \to 2\) has either an infinite 0-homogeneous set or a dense 1-homogeneous set.

Theorem (Pigeonhole principle on rationals)

\((\eta)^1_{<\infty}\) Let \(k \in \mathbb{N}\). Every coloring \(f : \mathbb{Q} \to k\) has a dense homogeneous set.
Theorem (Tree Theorem for pairs and two colors)

\[\mathsf{TT}_2^2\] Every coloring \( f : [2^\mathbb{N}]^2 \rightarrow 2 \) has a homogeneous tree.

Theorem (Pigeonhole Principle on trees)

\[\mathsf{TT}_1^1\] Let \( k \in \mathbb{N} \). Every coloring \( f : 2^\mathbb{N} \rightarrow k \) has a homogeneous tree.
Lemma (RCA₀)

- ACA₀ → (ℕ₀, η)² → RT²₂
- (ℕ₀, η)² → (η)¹₁
- IΣ₀² → (η)¹₁_<∞ → BΣ₀²
Lemma (RCA₀)

- ACA₀ → (\aleph_0, \eta)^2 → RT^2_2
- (\aleph_0, \eta)^2 → (\eta)^1_{<\infty}
- IΣ^0_2 → (\eta)^1_{<\infty} → BΣ^0_2

Theorem (F. and Patey)

- RCA₀ + BΣ^0_2 \not\vdash (\eta)^1_{<\infty}
- (\aleph_0, \eta)^2 \not\leq_c RT^2_{<\infty}
We separate $(\eta)^1_{<\infty}$ from $B\Sigma^0_2$ by adapting the model-theoretic proof of Corduan, Groszek, and Mileti that separates $TT^1$ from $B\Sigma^0_2$. 
We separate $(\eta)^{1}_{<\infty}$ from $B\Sigma^0_2$ by adapting the model-theoretic proof of Corduan, Groszek, and Mileti that separates $TT^1$ from $B\Sigma^0_2$.

Basically, in a model of $RCA_0 + \neg I\Sigma^0_2$, there is a real $X$ and an $X$-recursive instance of $(\eta)^{1}_{<\infty}$ with no $X$-recursive solutions.
The proof consists of two steps.

**Lemma (Step 1)**

*In a model $M$ of RCA$_0$, for every $X \in M$, there is a uniform $X$-recursive way, given finitely many $X$-r.e. subsets of $\mathbb{Q}$, to compute a 2-coloring $f : \mathbb{Q} \to 2$ so as to defeat all the given potential homogeneous sets.*
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**Lemma (Step 1)**

In a model $M$ of $\text{RCA}_0$, for every $X \in M$, there is a uniform $X$-recursive way, given finitely many $X$-r.e. subsets of $\mathbb{Q}$, to compute a 2-coloring $f : \mathbb{Q} \to 2$ so as to defeat all the given potential homogeneous sets.

To obtain such a result, we use a combinatorial feature of $(\eta)_1^{<\infty}$ shared by $\text{TT}^1$.

The basic idea is as follows. We are given many dense potential sets $W_e^X$ with $e < n$, and we build $f$ by stages.
The basic strategy to diagonalize against a single $W_e^X$ is to wait until we see 2 disjoint intervals with end-points in $W_e^X$ and then color the two intervals with 0 and 1 respectively. This works in isolation.
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We take care of all $W_e^X$’s by fixing $4n$ disjoint intervals with end-points in $W_e^X$ for every $W_e^X$ that outputs $4n + 1$ points (we say that $W_e^X$ requires attention). By a simple combinatorial argument, from $k \leq n$ tuples of $4n$ disjoint intervals we can select a pair from each tuple so as to have $2k$ disjoint intervals.
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At any stage we color every current pair of intervals with 0 and 1 respectively. Since there are finitely many $W_e^X$’s, we eventually stabilize on some pair for each $W_e^X$ that requires attention.
Lemma (Step 2)

Let $M$ be a model of $\text{RCA}_0$ and suppose that $M$ does not satisfy $I\Sigma^0_2(X)$ for some $X \subseteq M$. Then there is an $X$-recursive coloring $f$ of $\mathbb{Q}$ into finitely many colors such that no $X$-recursive dense set is homogeneous for $f$. 
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Let $M$ be a model of $\text{RCA}_0$ and suppose that $M$ does not satisfy $\Sigma^0_2(X)$ for some $X \subseteq M$. Then there is an $X$-recursive coloring $f$ of $\mathbb{Q}$ into finitely many colors such that no $X$-recursive dense set is homogeneous for $f$.

The failure of $\Sigma^0_2(X)$ implies that there is an $X$-recursive function $h: \mathbb{N}^2 \to \mathbb{N}$ such that for some number $a$, the range of the partial function $h(y) = \lim_{s \to \infty} h(y, s)$ is unbounded on $\{y : y < a\}$.
Theorem

Let $P$ be a $\Pi^1_1$ sentence. Then $\text{RCA}_0 + P \vdash (\eta)^1_{<\infty}$ if and only if $\text{RCA}_0 + P \vdash I\Sigma^0_2$. In particular, $\text{RCA}_0 + \text{B}\Sigma^0_2 \nvdash (\eta)^1_{<\infty}$. 
Theorem

Let $P$ be a $\Pi^1_1$ sentence. Then $\text{RCA}_0 + P \vdash (\eta)^1_{<\infty}$ if and only if $\text{RCA}_0 + P \vdash \text{IΣ}^0_2$. In particular, $\text{RCA}_0 + \text{BΣ}^0_2 \nvdash (\eta)^1_{<\infty}$.

Proof sketch.

Let $M$ be a model of $\text{RCA}_0 + P$ where $\text{IΣ}^0_2$ fails, and $X \in M$ as above. Then $\Delta^0_2(X)$ is a model of $\text{RCA}_0 + P$ where $(\eta)^1_{<\infty}$ fails.
Most implications of the form $Q \rightarrow P$ over RCA$_0$, where $P$ and $Q$ are $\Pi^1_2$ statements, make use only of one $Q$-instance to solve a $P$-instance. This is the notion of computable reducibility.

**Definition**

Fix two $\Pi^1_2$ statements $P$ and $Q$. $P$ is **computably reducible** to $Q$ (written $P \leq_c Q$) if every $P$-instance $I$ computes a $Q$-instance $J$ such that, for every solution $S$ to $J$, $I \oplus S$ computes a solution to $I$. 

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To show that $P \not\leq_c Q$, it is “enough” to produce a computable $P$-instance $I$ such that every computable $Q$-instance has a solution that does not compute a solution to $I$. 
$P \leq_c Q$ does not mean that $\text{RCA}_0 \vdash Q \rightarrow P$. In some cases, it is possible to obtain a separation over $\omega$-models from a one-step non-reduction.

- ADS does not imply CAC over $\text{RCA}_0$ (Lerman, Solomon, and Towsner)
- EM does not imply $\text{RT}^2_2$ over $\text{RCA}_0$ (Lerman, Solomon, and Towsner)
- $\text{RT}^2_2$ does not imply $\text{TT}^2_2$ over $\text{RCA}_0$ (Patey)

The above results use a general framework.

We prove that $(\mathbb{N}_0, \eta)^2 \not\leq_c \text{RT}^2_{<\infty}$. However, we are not able to generalize this result to a separation over $\omega$-models.
Why?

Basically, we want to produce an instance \( f : [\mathbb{Q}]^2 \to 2 \) of \((\omega_0, \eta)^2\) and solve instances of \( \text{RT}_2^2 \) without computing solutions to \( f \). We can view this as a game.
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Given an instance $g$ of $\text{RT}_2^2$ we are trying to build a solution $H$ to $g$ which does not compute a solution to $f$. We regard $f$ as our opponent. So, suppose we want to diagonalize against $\Phi_{g \oplus H}^0$ and $\Phi_{g \oplus H}^1$, where $\Phi_{i}^{g \oplus H}$ is a potential homogeneous set of color $i$. Our opponent $f$ commits to make $\Phi_{0}^{g \oplus H}$ infinite or $\Phi_{1}^{g \oplus H}$ dense.
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In the case of $\text{TT}_2^2$, our opponent commits to build a full binary tree in either case.
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In the case of $TT^2_2$, our opponent commits to build a full binary tree in either case.

This half commitment property is the main combinatorial difference between the two principles that prevents us from adapting the proof for $TT^2_2$. 
To show that $(\mathbb{N}_0, \eta)^2$ does not computably reduce to $\text{RT}^2_{<\infty}$, we consider the asymmetric version of $(\eta)^1_{<\infty}$.

$(\mathbb{N}_0, \eta)^1$ For every partition $A_0 \cup A_1 = \mathbb{Q}$ there is either an infinite subset of $A_0$ or a dense subset of $A_1$. 
To show that \((\mathbb{N}_0, \eta)^2\) does not computably reduce to \(\text{RT}^2_{<\infty}\), we consider the asymmetric version of \((\eta)^1_{<\infty}\).

\((\mathbb{N}_0, \eta)^1\) For every partition \(A_0 \cup A_1 = \mathbb{Q}\) there is either an infinite subset of \(A_0\) or a dense subset of \(A_1\).

**Theorem (F. and Patey)**

*There is a \(\Delta^0_2\) instance \(A_0 \cup A_1 = \mathbb{Q}\) of \((\mathbb{N}_0, \eta)^1\) such that every computable coloring \(g: [\omega]^2 \to k\) has an infinite homogeneous set \(H\) that does not compute a solution to \(A_0 \cup A_1 = \mathbb{Q}\).*
Corollary

There is a computable coloring $f : [\mathbb{Q}]^2 \to 2$ such that every computable coloring $g : [\omega]^2 \to k$ has an infinite homogeneous set $H$ that does not compute a solution to $f$.

Proof.
Let $f(x, s)$ be such that $f(x) = \lim_s f(x, s)$ exists and $x \in A_f(x)$.
The fairness notion

We design a **fairness property** for instances $A_0 \cup A_1 = \mathbb{Q}$ of $(\mathbb{N}_0, \eta)^1$. 
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Again, we see an instance of $(\mathbb{N}_0, \eta)^1$ as our opponent. The opponent is **fair** in the sense that if we have infinitely many chances to diagonalize against it, then it will allow us to do it.
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More precisely:

**($F$)** Given $f : [\omega]^2 \to k$, we are able to build infinite homogeneous sets $G_0, \ldots, G_{k-1}$, where $G_i$ is homogeneous with color $i$, such that for all $k$-tuples of Turing functionals $\Phi_0, \ldots, \Phi_{k-1}$, if every $\Phi_i^{G_i}$ is **large**, then one of them is not a solution to $A_0 \cup A_1 = \mathbb{Q}$.
The fairness notion for $(\aleph_0, \eta)^2$ is very technical. In general, it depends on the combinatorics of the problem (see CAC and $\text{TT}_2^2$).

- If an instance $A_0 \cup A_1 = \mathbb{Q}$ of $(\aleph_0, \eta)^1$ is fair with respect to a Scott set $S$ of reals ($\mathcal{F}$ holds for every $f \in S$), then every instance $f \in S$ of $\text{RT}^2_{\infty}$ has a solution that compute neither an infinite subset of $A_0$ nor a dense subset of $A_1$.

- The solutions to instances of $\text{RT}^2_{\infty}$ are built by using Mathias forcing over Scott sets.

- We can produce a $\Delta^2_0$ instance of $(\aleph_0, \eta)^1$ as above.
Questions

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*Does* \( (\aleph_0, \eta)^2 \) *imply* ACA\(_0\) *over* RCA\(_0\) *?*

Seetapun's argument does not work for \((\aleph_0, \eta)^2\). Actually, there is no forcing notion to build solutions to any instance of \((\aleph_0, \eta)^2\).
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Question

Does RT\(^2_2\) imply \((\aleph_0, \eta)^2\) over RCA\(_0\)?
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Question

Does RT₂⁻¹ imply $(\aleph_0, \eta)^2$ over RCA₀?

Question

Does $(\eta)^1_{\leq \infty}$ imply $I\Sigma^0_2$ over RCA₀?
References

Thanks for your attention