Some Progress on Kierstead’s Conjecture

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Computable Linear Orderings

\((L, \leq)\) is a computable linear ordering, if \(\leq\) is a linear ordering on \(L\) and both \(L\) and \(\leq\) are computable.

Some order types:

- \(\omega\);
- \(\omega^*\);
- \(\eta\);
- \(\zeta\);
- addition and product.

Folklore

There is a computable linear ordering \(L\) of order type \(\omega\) with \(S(x)\), the successor function, not computable.

References:

- **Computability Theory and Linear Orderings**, Rod Downey, Chapter 14 in “Handbook of Recursive Mathematics”.
About $\eta$

Folklore

$\eta$ is computably categorical (or autostable).
About \( \eta \)

Folklore
\( \eta \) is computably categorical (or autostable).

Remmel’s Characterization
A computable linear ordering \((L, \leq)\) is computably categorical if and only if it has only finitely many successivities.
A classical result
Any infinite linear ordering has an infinite subordering of order type either $\omega$ or $\omega^*$.

Theorem (Tennenbaum, Denisov)
There is a computable linear ordering of order type $\omega + \omega^*$ with no infinite computably enumerable suborderings of order type $\omega$ or $\omega^*$. 
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View from reverse math.
More on Effective considerations

Theorem (Rosenstein)
If \((L, \leq)\) is a computable linear ordering, then it has a computable subordering of type \(\omega, \omega^*, \omega + \omega^* \) or \(\omega + \zeta \eta + \omega^*\).
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**Theorem (Rosenstein)**

If $(L, \leq)$ is a computable linear ordering, then it has a computable subordering of type $\omega, \omega^*, \omega + \omega^*$ or $\omega + \zeta \eta + \omega^*$.

Rosenstein asks whether $\omega + \zeta \eta + \omega^*$ is necessary.
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Theorem (Lerman)
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**Theorem (Lerman)**
There is a computable linear ordering with no computable subordering of type \(\omega, \omega^*, \text{ or } \omega + \omega^*\).

**Theorem (Manaster)**
If \((L, \leq)\) is an infinite computable linear ordering, then \(L\) has a \(\Pi_1\) subset of type \(\omega\) or \(\omega^*\).
Self-embeddings

Dushnik-Miller Theorem
Every countable infinite linear ordering has a nontrivial self-embedding.
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Every countable infinite linear ordering has a nontrivial self-embedding.

**Theorem (Hay and Rosenstein)**
There is a computable linear ordering with no nontrivial computable self-embedding.

**Theorem (Downey and Lempp)**
There is a computable linear ordering \((L, \leq)\) such that if \(f\) is a nontrivial self-embedding of \(L\), then \(f\) computes \(\emptyset'\).

Dushnik-Miller Theorem is equivalent to ACA\(_0\) over RCA\(_0\).

**Theorem (Downey, Jockusch and Miller)**
There is a computable linear ordering of order type \(\omega\) with no nontrivial \(0'\)-computable self-embedding.
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Dushnik-Miller Theorem is equivalent to \(ACA_0\) over \(RCA_0\).

**Theorem (Downey, Jockusch and Miller)**
There is a computable linear ordering of order type \(\omega\) with no nontrivial \(\emptyset'\)-computable self-embedding.
A linear ordering is computably rigid if it has no nontrivial computable automorphisms.

**Theorem (Schwartz)**
A computable linear ordering has a computably rigid copy if and only if it has no *interval* of order type $\eta$. 
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**Theorem (Kierstead)**
There is a computable linear ordering of type $2 \cdot \eta$ with no nontrivial $\Pi_1$ automorphism.
Kierstead’s conjecture

Definition (Kierstead)
An automorphism is fairly trivial if for all \( x \), \([x, f(x)]\) is finite.
An automorphism is strongly nontrivial a nontrivial automorphism is not fairly trivial.

Kierstead’s Conjecture
For a computable linear ordering \( \mathcal{L} \), every computable copy of \( \mathcal{L} \) has a strongly nontrivial \( \Pi_1 \) automorphism if and only if the corresponding order type contains an interval of order type \( \eta \).
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This conjecture is true for \( 2 \cdot \eta \). For this case, there is no difference between “strongly nontrivial” and “nontrivial”.
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This conjecture is true for $2 \cdot \eta$. For this case, there is no difference between “strongly nontrivial” and “nontrivial”.

Downey and Moses proved that it is also true for discrete computable linear orderings.

Here a linear ordering is discrete if every element has both an immediate predecessor and an immediate successor, except for the possible first and last elements.
η-like

A linear ordering $\mathcal{L}$ is $\eta$-like if $\mathcal{L}$ is isomorphic to

$$\sum_{q \in \mathbb{Q}} F(q),$$

where $F$ is a function from $\mathbb{Q}$ to $\mathbb{N}\setminus\{0\}$.

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- $2 \cdot \eta$ is $\eta$-like.

**Theorem (Harris, Lee and Cooper)**

Suppose that $F : \mathbb{Q} \to \mathbb{N}\setminus\{0\}$ is $\emptyset'$-limitwise monotonic and that the linear ordering $\mathcal{L} \simeq \sum_{q \in \mathbb{Q}} F(q)$ has no dense intervals. Then $\mathcal{L}$ has a computable copy with no (strongly) nontrivial $\Pi_1$-automorphisms.

- This theorem improves Kierstead’s result a lot.
Extended $\emptyset'$-limitwise monotonic function

A function $F : \mathbb{Q} \rightarrow (\mathbb{N} \setminus \{0\}) \cup \{\zeta\}$ is an extended $\emptyset'$-limitwise monotonic function if we assume $\zeta > n$ for each $n \in \mathbb{N}$ and there is a $0'$-limitwise monotonic function $f : \mathbb{Q} \times \mathbb{N} \rightarrow (\mathbb{N} \setminus \{0\}) \cup \{\zeta\}$ such that

1. for all $q \in \mathbb{Q}, s \in \mathbb{N}$, $f(q, s) \leq f(q, s + 1)$;
2. for all $q \in \mathbb{Q}$, $\lim_{s \to \infty} f(q, s) = F(q)$;
3. if $\lim_{s \to \infty} f(q, s) = \zeta$, then there is an $s_0$ such that for all $s \geq s_0$, $f(q, s) = \zeta$.

For an extended $\emptyset'$-limitwise monotonic function $F$, we define linear ordering $\sum_{q \in \mathbb{Q}} F(q)$. 
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- This notion extends the one considered by Harris, Lee and Cooper, and maybe by Turetsky and Kach.

- $2 \cdot \eta + \zeta + 3 \cdot \eta$, $\zeta \cdot \eta$ are in our consideration, but not $\zeta \cdot \omega$. 
Almost trivial automorphisms

An automorphism $f$ of a linear ordering $\mathcal{L} = (L, \leq)$ is almost trivial if

$$(\forall x)[|\llbracket x \rrbracket_\mathcal{L}| > 1 \rightarrow f(\llbracket x \rrbracket_\mathcal{L}) = \llbracket x \rrbracket_\mathcal{L}].$$

- For discrete linear orderings, there is no difference between “fairly trivial” and “almost trivial”.

Theorem (Wu and Zubkov)

Suppose that $F$ is an extended $\emptyset'$-limitwise monotonic function and that the linear ordering $L \simeq \sum_{q \in Q} F(q)$ has no dense intervals. Then $L$ has a computable copy with only almost trivial $\Pi_1$-automorphisms.

This generalizes Harris-Lee-Cooper's result, and covers some instances of Downey-Moses' result.
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Thanks!