# Cofinality of classes of ideals with respect to Katětov and Katětov-Blass orders

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Section 1

# Ideals and Katětov(-Blass) order

# Ideals over a countable set

Let $X$ be a countable infinite set.
We say that $\mathcal{I}$ is an ideal over $X$ if $\mathcal{I}$ is a family of subsets of $X$ such that

- $A \subseteq B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$,
- $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$,
- $X \notin \mathcal{I}$,
- $\mathcal{I}$ contains all finite subsets of $X$.

- An ideal over a countable set $X$ can be identified with an ideal over $\omega$. We mainly discuss ideals over $\omega$.
- An ideal over a countable set $X$ is a subset of $\mathcal{P}(X)$, and $\mathcal{P}(X)$ can be naturally identified with the Cantor space $2^\omega$.

An ideal $\mathcal{I}$ over a countable set $X$ is said to be $\mathbf{\Sigma}^0_\xi$, $\mathbf{\Pi}^0_\xi$, Borel, $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, ... if it is $\mathbf{\Sigma}^0_\xi$, $\mathbf{\Pi}^0_\xi$, Borel, $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, ... as a subset of the Cantor space, respectively.

An ideal $\mathcal{I}$ is called a *P-ideal* if for any $\{A_n \mid n < \omega\} \subseteq \mathcal{I}$ there is $A \in \mathcal{I}$ s.t. $A_n \subseteq^* A$, i.e. $A_n \setminus A$ is finite, for all $n < \omega$.

# Some examples of ideals

1. The family of all finite subsets of $\omega$ is a $\mathbf{\Sigma}^0_2$ P-ideal over $\omega$. This ideal is denoted as $\mathrm{FIN}$.
2. For a function $f : \omega \to \mathbb{R}_{\geq 0}$ with $\sum_{n \in \omega} f(n) = \infty$,

$$\mathcal{I}_f := \{A \subseteq \omega \mid \sum_{n \in A} f(n) < \infty\}$$

   is a $\mathbf{\Sigma}^0_2$ P-ideal over $\omega$. $\mathcal{I}_f$ is called a *summable ideal* corresponding to $f$.
3. *The asymptotic density 0 ideal*

$$\mathcal{Z}_0 := \left\{ A \subseteq \omega \,\middle|\, \lim_{n \to \omega} \frac{|A \cap n|}{n} = 0 \right\}.$$

   is a $\mathbf{\Pi}^0_3$ P-ideal over $\omega$.
4. *The eventually different ideal*

$$\mathcal{ED} := \{A \subseteq \omega \times \omega \mid \exists m \in \omega \forall^\infty n,\ |A_{(n)}| < m\}$$

   is a $\mathbf{\Sigma}^0_2$ ideal over $\omega \times \omega$. ($A_{(n)} = \{k \mid (n, k) \in A\}$.)

# Orders on ideals

Let $X$, $Y$ be ctble. infinite sets, and let $\mathcal{I}, \mathcal{J}$ be ideals over $X$, $Y$, respectively.

- (Rudin-Keisler order)
  $\mathcal{I} \leq_{\mathrm{RK}} \mathcal{J}$ if there is $f : Y \to X$ such that for any $A \subseteq X$,
  $A \in \mathcal{I} \;\Leftrightarrow\; f^{-1}[A] \in \mathcal{J}$.
- (Rudin-Blass order)
  $\mathcal{I} \leq_{\mathrm{RB}} \mathcal{J}$ if there is a finite to one $f : Y \to X$ such that for any $A \subseteq X$,
  $A \in \mathcal{I} \;\Leftrightarrow\; f^{-1}[A] \in \mathcal{J}$.
- (Katětov order)
  $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$ if there is $f : Y \to X$ such that for any $A \subseteq X$,
  $A \in \mathcal{I} \Rightarrow f^{-1}[A] \in \mathcal{J}$.
- (Katětov-Blass order)
  $\mathcal{I} \leq_{\mathrm{KB}} \mathcal{J}$ if there is a finite to one $f : Y \to X$ such that for any $A \subseteq X$,
  $A \in \mathcal{I} \Rightarrow f^{-1}[A] \in \mathcal{J}$.

$$\begin{array}{ccc} \mathcal{I} \leq_{\mathrm{RB}} \mathcal{J} & \Longrightarrow & \mathcal{I} \leq_{\mathrm{RK}} \mathcal{J} \\ \Downarrow & & \Downarrow \\ \mathcal{I} \leq_{\mathrm{KB}} \mathcal{J} & \Longrightarrow & \mathcal{I} \leq_{\mathrm{K}} \mathcal{J} \end{array}$$

# Facts on Katětov and Katětov-Blass orders

- If $\mathcal{I} \subseteq \mathcal{J}$ are ideals over $X$, then $id_X$ witnesses that $\mathcal{I} \leq_{\mathrm{KB}} \mathcal{J}$.
- Many properties of ideals (or filters) can be characterized by the Katětov(-Blass) order and some Borel ideals. For example:
  - An ultrafilter $\mathcal{F}$ over $\omega$ is selective iff $\mathcal{ED} \not\leq_{\mathrm{K}} \mathcal{F}^*$.
  - An ultrafilter $\mathcal{F}$ over $\omega$ is P-point iff $\mathrm{FIN} \times \mathrm{FIN} \not\leq_{\mathrm{K}} \mathcal{F}^*$.
  - An ultrafilter $\mathcal{F}$ over $\omega$ is Q-point iff $\mathcal{ED}_{\mathrm{fin}} \not\leq_{\mathrm{KB}} \mathcal{F}^*$.
  - (Solecki) An ideal $\mathcal{I}$ over $\omega$ has the Fubini property iff $\mathcal{S} \not\leq_{\mathrm{K}} \mathcal{I} \restriction X$ for any $\mathcal{I}$-positive $X$.
- The Katětov order on Borel ideals is complicated:

**Theorem (Meza)**

$(\mathcal{P}(\omega)/\mathrm{FIN}, \subseteq^*)$ can be embeddable into (Borel ideals, $\leq_{\mathrm{K}}$).

Less is known about the structure of the Katětov and the Katětov-Blass orders on Borel ideals. In this talk we discuss these orders on the following classes of ideals:

- $\mathbf{\Sigma}^0_2$ ideals $\cdots$ the family of all $\mathbf{\Sigma}^0_2$ ideals.
- Borel ideals $\cdots$ the family of all Borel ideals over $\omega$.
- $\mathbf{\Sigma}^1_1$ ideals $\cdots$ the family of all $\mathbf{\Sigma}^1_1$ ideals.
- $\mathbf{\Sigma}^1_1$ P-ideals $\cdots$ the family of all $\mathbf{\Sigma}^1_1$ P-ideals.

## Fact

1. There is no $\mathbf{\Pi}^0_2$ ideal. So $\mathbf{\Sigma}^0_2$ ideals are the class of the simplest ideals.
2. (Solecki) Every $\mathbf{\Sigma}^1_1$ P-ideal is $\mathbf{\Pi}^0_3$.

$$\mathbf{\Sigma}^0_2 \text{ ideals},\ \mathbf{\Sigma}^1_1 \text{ P-ideals} \subsetneq \text{Borel ideals} \subsetneq \mathbf{\Sigma}^1_1 \text{ ideals}$$

We will show that all of these classes are upward directed and discuss their cofinal types.

Section 2

# Directedness

# Directedness

**Theorem**

$\mathbf{\Sigma}^0_2$ ideals, $\mathbf{\Sigma}^1_1$ P-ideals, Borel ideals and $\mathbf{\Sigma}^1_1$ ideals are all upward directed with respect to $\leq_{\mathrm{KB}}$. (So they are upward directed w.r.t. $\leq_{\mathrm{K}}$, too.)

We give an outline of the proof.

**Recall**

$\mathcal{I}$: an ideal over $X$, $\mathcal{J}$: an ideal over $Y$.

- (Katětov order)
  $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$ if there is $f : Y \to X$ such that for any $A \subseteq X$,
  $A \in \mathcal{I} \Rightarrow f^{-1}[A] \in \mathcal{J}$.
- (Katětov-Blass order)
  $\mathcal{I} \leq_{\mathrm{KB}} \mathcal{J}$ if there is a finite to one $f : Y \to X$ such that for any $A \subseteq X$,
  $A \in \mathcal{I} \Rightarrow f^{-1}[A] \in \mathcal{J}$.

# $\boldsymbol{\Sigma}^1_1$ ideals

First we show the directedness of $\boldsymbol{\Sigma}^1_1$ ideals.

Before proving the directedness w.r.t. $\leq_{\mathrm{KB}}$, we observe the directedness w.r.t. $\leq_{\mathrm{K}}$:

- Suppose $\mathcal{I}_0$ and $\mathcal{I}_1$ are $\boldsymbol{\Sigma}^1_1$.
- For $k = 0, 1$ let $\pi_k : \omega \times \omega \to \omega$ be the $k$-th projection, i.e. $\pi_k(n_0, n_1) = n_k$. Let

$$\mathcal{J} \; := \; \{B \subseteq \omega \times \omega \mid \exists A_0 \in \mathcal{I}_0 \exists A_1 \in \mathcal{I}_1,\; B \subseteq {\pi_0}^{-1}[A_0] \cup {\pi_1}^{-1}[A_1]\} \,.$$

- It is easy to check that $\mathcal{J}$ is a $\boldsymbol{\Sigma}^1_1$ ideal over $\omega \times \omega$.

  Moreover $\pi_k$ witnesses that $\mathcal{I}_k \leq_{\mathrm{K}} \mathcal{J}$ for each $k = 0, 1$. $\square$

Note that $\pi_k$ is not finite to one. So this does not give the directedness w.r.t. $\leq_{\mathrm{KB}}$.

For the directedness of $\mathbf{\Sigma}^1_1$ ideals w.r.t. $\leq_{\mathrm{KB}}$, we use the following:

**Theorem (Mathias)**

For any $\mathbf{\Sigma}^1_1$ ideal $\mathcal{I}$ over $\omega$ it holds that $\mathrm{FIN} \leq_{\mathrm{RB}} \mathcal{I}$, that is, there is a finite to one $f : \omega \to \omega$ such that $f^{-1}[C] \in \mathcal{I}$ iff $C$ is finite.

Suppose $\mathcal{I}$ is a $\mathbf{\Sigma}^1_1$ ideal, and let $f : \omega \to \omega$ be as above.

Let $\langle k_m \mid m \in \omega \rangle$ be the increasing enumeration of the range of $f$, and let $X_m := f^{-1}(k_m)$. Then

- $\langle X_m \mid m \in \omega \rangle$ is a partition of $\omega$ into finite sets,
- For any $A \in \mathcal{I}$ the set $M = \{m \mid X_m \subseteq A\}$ is finite.

  (Otherwise, $C = \{k_m \mid m \in M\}$ is infinite, but $f^{-1}[C] = \bigcup_{m \in M} X_m \subseteq A \in \mathcal{I}$.)

## Directedness of $\mathbf{\Sigma}^1_1$ ideals w.r.t. $\leq_{\mathrm{KB}}$

- Suppose $\mathcal{I}_0$ and $\mathcal{I}_1$ are $\mathbf{\Sigma}^1_1$ ideals over $\omega$.
- For $k = 0, 1$ let $\langle X^k_m \mid m < \omega \rangle$ be a partition of $\omega$ into finite sets such that for any $A \in \mathcal{I}_k$ there are at most finitely many $m$ with $X_m \subseteq A$.
- Let $X := \bigcup_{m \in \omega} X^0_m \times X^1_m \subseteq \omega \times \omega$. Let $\pi_k : X \to \omega$ be the $k$-th projection. Note that $\pi_k$ is finite to one.
- Let $\mathcal{J} := \{B \subseteq X \mid \exists A_0 \in \mathcal{I}_0 \exists A_1 \in \mathcal{I}_1,\ B \subseteq {\pi_0}^{-1}[A_0] \cup {\pi_1}^{-1}[A_1]\}$.
- $\mathcal{J}$ is a $\mathbf{\Sigma}^1_1$ ideal over $X$.

  Proof of $X \notin \mathcal{J}$

  Suppose $A_0 \in \mathcal{I}_0$ and $A_1 \in \mathcal{I}_1$. There is $m \in \omega$ s.t. $X^0_m \not\subseteq A_0$ and $X^1_m \not\subseteq A_1$. Then $X^0_m \times X^1_m \not\subseteq {\pi_0}^{-1}[A_0] \cup {\pi_1}^{-1}[A_1]$. So $X \not\subseteq {\pi_0}^{-1}[A_0] \cup {\pi_1}^{-1}[A_1]$.
- Clearly $\pi_k$ witnesses that $\mathcal{I}_k \leq_{\mathrm{KB}} \mathcal{J}$ for each $k = 0, 1$. □

# Directedness of other classes w.r.t. $\leq_{\mathrm{KB}}$

$\mathbf{\Sigma}^0_2$ ideals

In the proof for $\mathbf{\Sigma}^1_1$ ideals, if $\mathcal{I}_0$ and $\mathcal{I}_1$ are $\mathbf{\Sigma}^0_2$, then so is $\mathcal{J}$.
This follows from the compactness of the Cantor space. (Continuous images of $\mathbf{\Sigma}^0_2$ sets are $\mathbf{\Sigma}^0_2$.)

$\mathbf{\Sigma}^1_1$ P-ideals

In the proof for $\mathbf{\Sigma}^1_1$ ideals, if $\mathcal{I}_0$ and $\mathcal{I}_1$ are P-ideals, then so is $\mathcal{J}$.

Borel ideals

Borel ideals are cofinal in $\mathbf{\Sigma}^1_1$ ideals w.r.t. $\leq_{\mathrm{KB}}$ by the following fact:

**Fact (folklore)**

For any $\mathbf{\Sigma}^1_1$ ideal $\mathcal{I}$ there is a Borel ideal $\mathcal{J}$ with $\mathcal{I} \subseteq \mathcal{J}$.

# Question

I do not know whether other classes are directed:

**Qestion**

For $\alpha > 2$, are $\mathbf{\Sigma}^0_\alpha$ ideals directed with respect to $\leq_{\mathrm{KB}}$ (or $\leq_{\mathrm{K}}$) ?

Section 3

# Cofinal types

# Tukey order

Let $\mathcal{D} = (D, \leq_D)$ and $\mathcal{E} = (E, \leq_E)$ be (upward) directed sets.

- $\mathcal{D} \leq_{\mathrm{T}} \mathcal{E}$ if there is a function $f : E \to D$ such that images of cofinal subsets of $\mathcal{E}$ are cofinal in $\mathcal{D}$.
- $\mathcal{D} \equiv_{\mathrm{T}} \mathcal{E}$ if $\mathcal{D} \leq_{\mathrm{T}} \mathcal{E}$, and $\mathcal{E} \leq_{\mathrm{T}} \mathcal{D}$.

If $\mathcal{D} \equiv_{\mathrm{T}} \mathcal{E}$, then we say that the cofinal types of $\mathcal{D}$ and $\mathcal{E}$ are the same.

- $\mathcal{D} \leq_{\mathrm{T}} \mathcal{E}$ iff there is a function $g : D \to E$ such that images of unbounded subsets of $\mathcal{D}$ are unbounded in $\mathcal{E}$.

For a directed set $\mathcal{D} = (D, \leq_D)$ let

$$\begin{aligned} \mathrm{cof}(\mathcal{D}) &:= \min\{|A| \mid A \text{ is a cofinal subset of } \mathcal{D}\}, \\ \mathrm{ubdd}(\mathcal{D}) &:= \min\{|A| \mid A \text{ is an unbounded subset of } \mathcal{D}\}. \end{aligned}$$

- If $\mathcal{D} \equiv_{\mathrm{T}} \mathcal{E}$, then $\mathrm{cof}(\mathcal{D}) = \mathrm{cof}(\mathcal{E})$, and $\mathrm{ubdd}(\mathcal{D}) = \mathrm{ubdd}(\mathcal{E})$.

# Cofinal types of $\mathbf{\Sigma}^0_2$ ideals and $\mathbf{\Sigma}^1_1$ P-ideals

It is known, due to Mazur and Solecki, that $\mathbf{\Sigma}^0_2$ ideals and $\mathbf{\Sigma}^1_1$ P-ideals have nice characterizations using lower semi-continuous submeasures.
Using these characterizations, we can prove the following:

**Theorem (Minami-S.)**

1. $(\mathbf{\Sigma}^0_2 \text{ ideals}, \leq_{\mathrm{K}}) \equiv_{\mathrm{T}} (\mathbf{\Sigma}^0_2 \text{ ideals}, \leq_{\mathrm{KB}}) \equiv_{\mathrm{T}} (\omega^\omega, \leq^*)$,
   where for $f, g \in \omega^\omega$, $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$.
2. The family of all summable ideals are unbounded in $(\mathbf{\Sigma}^0_2 \text{ ideals}, \leq_{\mathrm{KB}})$.

**Corollary (Minami-S.)**

- $\mathrm{cof}(\mathbf{\Sigma}^0_2 \text{ ideals}, \leq_{\mathrm{K}}) = \mathrm{cof}(\mathbf{\Sigma}^0_2 \text{ ideals}, \leq_{\mathrm{KB}}) = \mathfrak{d}$.
- $\mathrm{ubdd}(\mathbf{\Sigma}^0_2 \text{ ideals}, \leq_{\mathrm{K}}) = \mathrm{ubdd}(\mathbf{\Sigma}^0_2 \text{ ideals}, \leq_{\mathrm{KB}}) = \mathfrak{b}$.

**Theorem (Minami-S.)**

$(\mathbf{\Sigma}^1_1 \text{ P-ideals}, \leq_{\mathrm{KB}})$ has the greatest element. (Hence so does $(\mathbf{\Sigma}^1_1 \text{ P-ideals}, \leq_{\mathrm{K}})$.)

# Characterizations by lower semi-continuous submeasures

A *lower semi-continuous submeasure* (l.s.c.s.) on $\omega$ is a function $\varphi : \mathcal{P}(\omega) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that

- $\varphi(\emptyset) = 0$,
- $A \subseteq B \;\Rightarrow\; \varphi(A) \leq \varphi(B)$,
- $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$,
- $\varphi(A) = \lim_{n\to\omega} \varphi(A \cap n)$. (Lower semi-continuity)

## Fact (1: Mazur, 2: Solecki)

1. $\mathcal{I}$ is a $\mathbf{\Sigma}^0_2$ ideal over $\omega$ iff there is a l.s.c.s. $\varphi$ with $\varphi(\omega) = \infty$ s.t.

$$\mathcal{I} \;=\; \{A \subseteq \omega \mid \varphi(A) < \infty\}.$$

2. $\mathcal{I}$ is a $\mathbf{\Sigma}^1_1$ P-ideal over $\omega$ iff there is a l.s.c.s. $\varphi$ with $\lim_{n\to\omega} \varphi(\omega \setminus n) > 0$ s.t.

$$\mathcal{I} \;=\; \{A \subseteq \omega \mid \lim_{n\to\omega} \varphi(A \setminus n) = 0\}.$$

# Points of proofs of theorems

- Each l.s.c.s. can be seen as a limit of its finite initial segments, which are submeasures on finite sets. Moreover we may assume that finite initial segments take values in $\mathbb{Q}$. (So the variation of finite initial segments are countable.)
- Between submeasures on finite sets we can define a directed order, which approximates the Katětov(-Blass) order between ideals obtained by l.s.c.s.'s

# Cofinal types of Borel ideals and $\boldsymbol{\Sigma}^1_1$ ideals

- Because Borel ideals are cofinal in $\boldsymbol{\Sigma}^1_1$ ideals w.r.t. $\subseteq$,
  - (Borel ideals, $\leq_{\mathrm{K}}$) $\equiv_{\mathrm{T}}$ ($\boldsymbol{\Sigma}^1_1$ ideals, $\leq_{\mathrm{K}}$),
  - (Borel ideals, $\leq_{\mathrm{KB}}$) $\equiv_{\mathrm{T}}$ ($\boldsymbol{\Sigma}^1_1$ ideals, $\leq_{\mathrm{KB}}$),

  I do not know the cofinal types of them.
- It is known that these do not have the greatest element.
- If the following question is true, then the cofinal type of these are the same as $(\omega_1, <)$:

**Question**

For any $\alpha < \omega_1$ are $\boldsymbol{\Sigma}^0_\alpha$ ideals bounded in (Borel ideals, $\leq_{\mathrm{KB}}$) ?

- The above question is true for $\alpha = 2$:

  Using the characterization by l.s.c.s., it can be proved that for any $\boldsymbol{\Sigma}^0_2$ ideal $\mathcal{I}$ there is a $\boldsymbol{\Sigma}^1_1$ P-ideal $\mathcal{J}$ with $\mathcal{I} \subseteq \mathcal{J}$ (so $\mathcal{I} \leq_{\mathrm{KB}} \mathcal{J}$). Then the greatest element of ($\boldsymbol{\Sigma}^1_1$ P-ideals, $\leq_{\mathrm{KB}}$) is an upper bound of $\boldsymbol{\Sigma}^0_2$ ideals w.r.t. $\leq_{\mathrm{KB}}$.