An introduction to Special Session on Professor Tanaka’s Works
(His works and me)

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Prof. Tanaka’s Personal History

1955  Born in Tokyo
1978  Bachelor in Tokyo Institute of Technology
1980  Master in Tokyo Institute of Technology
1986  Ph.D in University of California, Berkeley (supervisor: Leo A. Harrington)
1986  Research Associate in Tokyo Institute of Technology
1991  Associate Professor in Tohoku University
1997  Professor in Tohoku University
Works

He has published more than 40 papers and 11 books (including 4 translations) and edited a serieses.

His articles can be mainly classified into 4 categories according to subjects:

1. Models of Arithmetics (12%): T. Wong
2. Complexity of diverse computational objects (And-Or trees etc.) (17%): T. Suzuki
3. Reverse Mathematics on Infinite Games (25%) : K. Yoshii
4. Others: Reverse Mathematics etc
Memories of his first works

In 1997, Tanaka published his self-embedding theorem in ”The self-embedding theorem of WKL\textsubscript{0} and a non-standard method” in Annals of Pure and Applied Logic 84.

**Theorem 1** (the self-embedding theorem). Let $(M, S)$ be a countable non-standard model of WKL\textsubscript{0}. Then there exists a proper initial segment $I \subseteq M$ such that $(M, S)$ is isomorphic to $(I, S[I])$, where $S[I] = \{X \cap I : X \in S\}$.

Using the theorem and exchanging the role of $(M, S)$ and the initial part, we can say that any countable non-standard model of WKL\textsubscript{0} has a “non-standard universe”. We can declare that this theorem and its application is one of his most innovative results.
By this method, he gave non-standard proofs of theorems derived from \( \text{WKL}_0 \) one after another: the maximum principle, Riemann integrability, Peano existence’s theorem etc.

Along a series of these results, Tanaka and I showed

**Theorem 2** (Tanaka and Y. 00). *The following are equivalent to each other over \( \text{RCA}_0 \)

1. \( \text{WKL}_0 \)

2. *Any compact group \( G \) has a Haar measure, more prescicely, a \( \sigma \)-additive left-invariant positive linear operator \( \mu \) on the space \( C(G) \) such that \( \mu(1) = 1 \).*

**Memories:** At that time, he tried to formalize and modify many kinds of its proof. When he found a non-standard proof by Hauser and others, he must have been certain of our success. A simple idea told us that his prospect is true.
Another important result owed to him is the following conservativity of WKL\(_0\).

**Theorem 3** (Simpson, Tanaka and Y. 02). *If WKL\(_0\) proves \(\forall X\exists! Y \varphi(X, Y)\) with \(\varphi\) arithmetical, so does RCA\(_0\).*

**Memories:** This statement was called Tanaka’s conjecture. To show the conjecture, we tried various approaches and tried to find evidences. After we repeated trial and error, the conjecture was solved by Prof. Simpson’s essential contribution in the end.
Tanaka’s conjecture is not unrelated to the previous theorem. “the unique existence of Haar measure” looks of the form in Theorem 3. (Indeed it is not correct.)

With RCA₀, we prove that the real number system ℝ satisfies the axioms of real closed ordered fields RCOF. However, this fact does not necessarily mean that the system ℝ satisfies all the theorems of RCOF.

Simpson and Tanaka claimed with no details of the proof that within RCA₀, ℝ does in fact satisfy the theorems of RCOF, with an appropriate notion of satisfaction. Then we basically adopt the definition of satisfaction relation Satₚ, which is given by quantifier elimination.

**Theorem 4** (Tanaka and Y. 05). *Any provably total function in RCOF is also total in ℝ via Satₚ.*

Later Tanaka and Sakamoto completed the proof.
Recent

Reverse mathematics on Brouwer’s fixed-point theorem is also one of Tanaka’s early result.

**Theorem 5** (Shioji-Tanaka, 90). *Brouwer fixed point theorem: “Let $C$ be the convex hull of a non-empty finite subsets of $\mathbb{R}^n$. Then every continuous function $f : C \to C$ has a fixed point”*, is equivalent to $\text{WKL}_0$ over $\text{RCA}_0$.

Recently, using this result, W. Peng and I showed:

1. A variant of Browder’s fixed point theorem is equivalent to $\text{WKL}_0$ over $\text{RCA}_0$.

2. A variant of Kakutani’s fixed point theorem is equivalent to $\text{ACA}_0$ over $\text{RCA}_0$.

3. …
Supplement I

According to the number of papers and the period of research, his most important result is to check the proof theoretic strength for determinacy of many classes of infinite games.

Moreover, many students of him studied this topic.


4. ...
Although I am greatly influenced by him and I am a coauthor of 9 papers, I have not studied determinacy.

Recently I became interested in determinacy of Blackwell games and its extensions.

Then, I may have a chance to have my name entered in the list in the near future.
Supplement II

Professor Tanaka is also an excellent educator and populaizer!!

1. He has guided about 40 students (of course, including me!)

2. According to him, we form a big group.

3. He published 11 books (including 4 translations) and edited a serieses.
Dear Professor Tanaka,
Congratulations on your 60th birthday!!