A simple conservation proof for ADS

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Ascending descending sequence

Today's target:

Definition

ADS: every infinite linear ordering has an infinite ascending or descending sequence.

ADS is an easy consequence of RT^2_2.
In fact, we can easily see the following.

Theorem (Shore/Hirschfeldt 2007)

ADS *is equivalent to transitive* RT^2_2*, i.e., Ramsey's theorem for transitive colorings.*
(Here, $P : [\mathbb{N}]^2 \to 2$ *is said to be transitive if* $P(a,b) = P(b,c) \to P(a,b) = P(a,c)$*.)*

So, ADS is a restricted version of RT^2_2.

Main question

Question

What is the proof-theoretic strength, or provably total functions (in other words, Π^0_2-part) of ADS?

In fact, we already know the result.

Theorem (Chong/Slaman/Yang 2012)

$\mathrm{ADS} + \mathsf{WKL}_0$ *is a* Π^1_1*-conservative extension of* $\mathrm{B}\Sigma^0_2$.

Corollary ("Proof-theoretic proof" by Kreuzer 2012")

The Π^0_2*-part of* $\mathrm{ADS} + \mathsf{WKL}_0$ *is* PRA.

- The proof of the above theorem is very complicated.
- Careful checking is needed to know the consistency strength.

Today, we would like to give a simpler proof of this corollary.

Ramsey's theorem and its finite approximation

The Π^0_2-part of (infinite) Ramsey's theorem is characterized by iterated Paris-Harrington-like principles.

Definition (RCA_0)

- A finite set $X \subseteq \mathbb{N}$ is said to be 0-*dense*(n, k) if $|X| \geq \min X$.
- A finite set X is said to be $m+1$-*dense*(n, k) if for any $P : [X]^n \to k$, there exists $Y \subseteq X$ which is m-dense(n, k) and P-homogeneous.

Note that "X is m-dense(n, k)" can be expressed by a Σ^0_0-formula.

Definition

- $m\mathrm{PH}^n_k$: for any $a \in \mathbb{N}$ there exists an m-dense(n, k) set X such that $\min X > a$.

Paris's argument

By the usual indicator arguments introduced by Paris, the following is known.

Theorem (essentially due to Paris 1978)

$\mathsf{WKL}_0 + \mathrm{RT}^n_k$ *is a conservative extension of* $\mathrm{I}\Sigma_1 + \{m\mathrm{PH}^n_k \mid m \in \omega\}$ *with respect to* Π^0_2*-sentences.*

Note that similar arguments work for Π^0_3 and Π^0_4-part.
The above conservation proof is formalizable within WKL_0, and thus we have the following.

Theorem

Over $\mathrm{I}\Sigma_1$*,* $\forall m\, m\mathrm{PH}^n_k$ *is equivalent to the* Σ_1*-soundness of* $\mathsf{WKL}_0 + \mathrm{RT}^n_k$.

Note that a similar argument works with a weaker base system RCA^*_0.

ADS and its finite approximation

Since ADS is equivalent to the transitive Ramsey's theorem, its Π^0_2-part is characterized by the same arguments.

Definition (RCA_0)

- A finite set $X \subseteq \mathbb{N}$ is said to be 0-*dense for* ADS if $|X| \geq \min X$.
- A finite set X is said to be $m+1$-*dense for* ADS if for any transitive $P : [X]^2 \to 2$, there exists $Y \subseteq X$ which is m-dense for ADS and P-homogeneous.

Definition

- $m\mathrm{PH}^{\mathrm{ADS}}$: for any $a \in \mathbb{N}$ there exists an m-dense for ADS set X such that $\min X > a$.

Paris's argument for ADS

Theorem

$\mathsf{WKL}_0 + \mathrm{ADS}$ *is a conservative extension of* $\mathrm{I}\Sigma_1 + \{m\mathrm{PH}^{\mathrm{ADS}} \mid m \in \omega\}$ *with respect to* Π^0_2*-sentences.*

The above conservation proof is again formalizable within WKL_0, and thus we have the following.

Theorem

Over $\mathrm{I}\Sigma_1$, $\forall m\, m\mathrm{PH}^{\mathrm{ADS}}$ *is equivalent to the* Σ_1*-soundness of* $\mathsf{WKL}_0 + \mathrm{ADS}$.

What we need to know is $m\mathrm{PH}^{\mathrm{ADS}}$.

α-large sets

We want to calculate the size of m-dense set for ADS.
We use a tool from proof theory.

Definition

For ordinals below ω^ω (with a fixed primitive recursive ordinal notation),

- X is said to be $\alpha + 1$-large if $X - \{\min X\}$ is α-large,
- X is said to be γ-large if X is $\gamma[\min X]$-large (γ: limit), where $\alpha + \omega^k[x] = \alpha + \omega^{k-1} \cdot x$.

- X is m-large if $|X| \geq m$.
- X is ω-large if $|X| \geq \min X$, *i.e.*, relatively large.
- X is ω^2-large if X splits up into $\min X$ many ω-large sets.
- ...

Density vs α-largeness

Here is a classical important result connecting α-largeness and PH-like statements.

Theorem (Solovay/Katonen 1981)

X is $\omega^{k+3} + \omega^3 + k + 4$-large $\Rightarrow$ X is 1-dense$(2, k)$.

Question

How big α is enough for the following?
X is α-large $\Rightarrow$ X is m-dense$(2, 2)$.

- An optimal answer to this question gives the proof-theoretic strength of RT^2_2, which is a famous open question in the field of reverse math.
- A naive approach only gives an upper bound ω_{m+1} for m-dense$(2, 2)$.

On the other hand, this approach works well for ADS.

Calculation

By S/K-theorem, ω^6-largeness is enough for 1-dense for ADS. Thus, X is 2-dense for ADS if it is large enough to find a ω^6-large solution.

Definition

X is said to be $(1, \alpha)$-*dense for* ADS if for any transitive $P : [X]^2 \to 2$, there exists $Y \subseteq X$ which is α-large and P-homogeneous.

Thanks to the transitivity, we can calculate the size of the above sets directly.

Lemma

X is 1-dense$(2, 2k) \Rightarrow X$ is $(1, \omega^k)$-dense for ADS.

Calculation

Now we can calculate the size of 2-dense sets.

- 2-dense for $\mathrm{ADS} \Leftarrow (1, \omega^6)$-dense for ADS
  $\Leftarrow$ 1-dense$(2, 12) \Leftarrow \omega^{16}$-large.

We can repeat this process.

- 3-dense for $\mathrm{ADS} \Leftarrow (1, \omega^{12})$-dense for ADS
  $\Leftarrow$ 1-dense$(2, 24) \Leftarrow \omega^{28}$-large.
- 4-dense for $\mathrm{ADS} \Leftarrow (1, \omega^{28})$-dense for ADS
  $\Leftarrow$ 1-dense$(2, 56) \Leftarrow \omega^{60}$-large.
- ...

Theorem

X is $\omega^{3^{m+1}}$-large $\Rightarrow$ X is m-dense for ADS.

ADS and its finite approximation (review)

Definition

- $m\mathrm{PH}^{\mathrm{ADS}}$: for any $a \in \mathbb{N}$ there exists an m-dense for ADS set X such that $\min X > a$.

Theorem

$\mathsf{WKL}_0 + \mathrm{ADS}$ is a conservative extension of $\mathrm{I}\Sigma_1 + \{m\mathrm{PH}^{\mathrm{ADS}} \mid m \in \omega\}$ with respect to Π^0_2-sentences.

Theorem

Over $\mathrm{I}\Sigma_1$, $\forall m\mathrm{PH}^{\mathrm{ADS}}$ is equivalent to the Σ_1-soundness of $\mathsf{WKL}_0 + \mathrm{ADS}$.

The strength of ADS

Lemma

For any $a \in \mathbb{N}$, $[a, F_m(a)]$ is a ω^m-large set.

Theorem

For any $m \in \omega$, PRA $\vdash m\mathrm{PH}^{\mathrm{ADS}}$.

Corollary

The Π^0_2-part of $\mathrm{ADS} + \mathrm{WKL}_0$ *is* $\mathrm{I}\Sigma_1$*, or equivalently,* PRA.

This conservation proof is easily formalizable within WKL_0. Thus, we have the following.

Corollary

$\mathrm{Con}(\mathrm{ADS} + \mathrm{WKL}_0)$ *is equivalent to* $\mathrm{Con}(\mathrm{PRA})$ *over* PRA.

Questions

Question

Is there a speed-up between $\mathrm{ADS} + \mathsf{WKL}_0$ and RCA_0?

A good lower bound for *m*-dense for ADS would give a positive answer.

And, again,

Question

how big α is enough for the following?
X is α-large $\Rightarrow$ X is m-dense$(2, 2)$.

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