A Survey of Determinacy of Infinite Games in Second Order Arithmetic,

dedicating to 60 ’ s birthday of Professor Tanaka.

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Introduction

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Focus on determinacy on Z_2

and see some related results.

Def. of Infinite Game

(i) Let $A \subseteq \mathbb{N}^{\mathbb{N}}$ be a set in $\mathcal{C}$.

(ii) Player I and II alternately choose natural numbers as follows:

$$\begin{array}{l|llll} \text{I} & n_0 & & n_2 & \dots \\ \hline \text{II} & & n_1 & & \dots \end{array}$$

(iii) I wins if $n_0, n_1, n_2 \cdots \in A$. II wins if not.

(iv) Game on A is determinate if one of the players has a winning strategy.

Notes

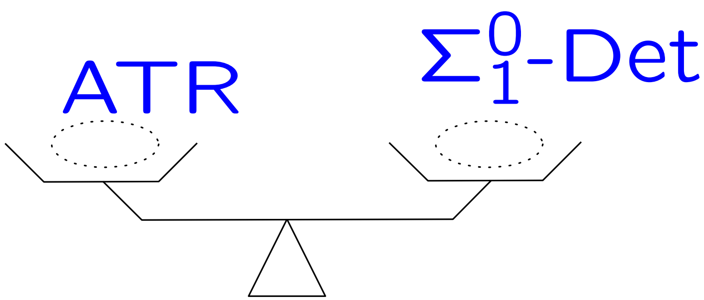
- Difficulties to prove the determinacy of game on A, (or, compute the winning strategy)

$\Rightarrow$ Depending on the complexities of set A. (e.g. A could be $\mathbf{\Delta}^0_1$ (clopen), $\mathbf{\Sigma}^0_1$ (open), $\mathbf{\Sigma}^0_2, \mathbf{\Delta}^0_3, \mathbf{\Pi}^0_3, \ldots \mathbf{\Delta}^1_1$ (Borel) ....

What determinacy asserts?

- Game G_A is determinate means that one of the players have a winning strategy.
- It asserts the existence of real number with certain complexity.

(e.g.)



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- $\mathsf{ZFC} \vdash$ Borel determinacy. (D. Martin, 1975)

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- $Z_2 \nvdash \Sigma^0_4$-Det. (D. Martin, 1974)
- ZFC$\vdash$ Borel determinacy. (D. Martin, 1975)
- And, in 1976 J. Steel showed that one of

the earliest results of Reverse Mathematics:

(RCA_0) ATR$\leftrightarrow \Sigma^0_1$-Det.

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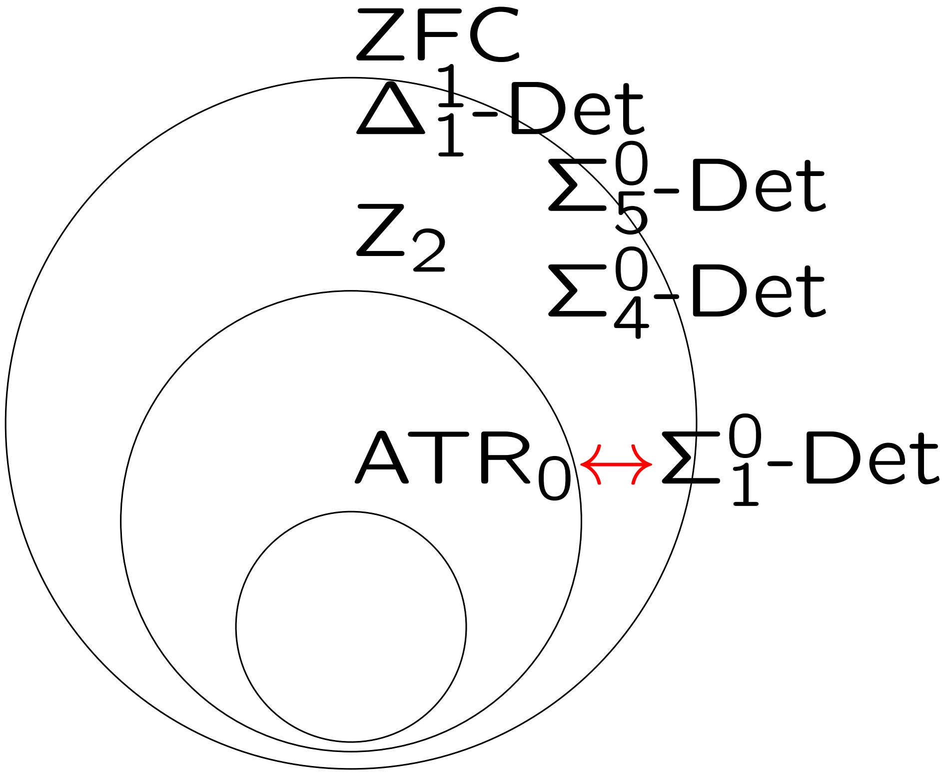

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Importances of those theorems

- Defference b/w Boldface and Lighrface version.

(Boldface) $\mathsf{RCA}_0 \vdash \Pi^1_1\text{-CA} \leftrightarrow \Sigma^0_1 \wedge \Pi^0_1\text{-Det}$.

Importances of those theorems (Lightface)

(i) $\mathsf{ACA}_0 \vdash (\Sigma^0_1 \wedge \Pi^0_1)$-Det $\rightarrow \Pi^1_1$-CA

(ii) $\mathsf{ATR}_0 \vdash \Pi^1_1$-CA $\rightarrow (\Sigma^0_1 \wedge \Pi^0_1)$-Det

Base theory of (ii) can not be weaker than ATR_0

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Base theory of (ii) can not be weaker than ATR_0

($\because$) Set of sure winning positions for player I can be constructed by ATR_0 with Π^1_1-oracle.

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(e.g.) In the proof of
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an infinite sequence of natural numbers satisfying Π^1_1-formula will be constructed in the game by players.

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- Letting players construct sets in the games.

(e.g.) In the proof of

"$\Sigma^0_1 \wedge \Pi^0_1$-Det $\to \Pi^1_1$-CA",

For any $\varphi(n)$: Π^1_1, there exists $\theta(n, X)$: Δ^0_1 s.t.

$$\varphi(n) \leftrightarrow \forall f \exists m \theta(n, f[m]).$$

Above ATR?

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Σ^1_1-ID_0, a new axiom system of Z_2

- It asserts the existence of inductively defined pre-well-ordering. (We will See next)
- Different with comprehension axioms as before. (Some reason?)
- Some Varieties of Σ^1_1-ID_0? (Future studies)

Def. of Σ^1_1-ID$_0$

$\triangleright$An operator Γ is a function from $\Gamma : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$.

$\triangleright$ If "$x \in \Gamma(X)$" is represented by a Σ^1_1 formula, then Γ is called Σ^1_1 operator.

● Σ^1_1-ID: for any Σ^1_1-operator Γ, there exists pre-wellordering $V \subset \mathbb{N} \times \mathbb{N}$ s.t. the following holds:

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● Σ^1_1-ID: for any Σ^1_1-operator Γ, there exists pre-wellordering $V \subset \mathbb{N} \times \mathbb{N}$ s.t. the following holds:

▷$\forall x \in F (V_x = \Gamma(V_{<x}) \cup V_{<x})$,

▷$\Gamma(F) \subset F$.

where

$V_x = \{y \in F : y \leq_V x\}$,

$V_{<x} = \{y \in F : y <_V x\}$, $F = \{x : x \leq_V x\}$.

Image of Σ^1_1-ID

▷ Apply Γ to $\emptyset$

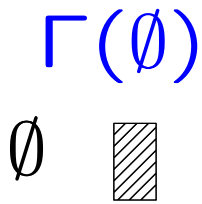

Image of Σ^1_1-ID

▷ Apply Γ to $\Gamma(\emptyset)$ and take the union:

$$\Gamma(\emptyset) \cup \Gamma(\Gamma(\emptyset))$$

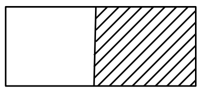

Image of Σ^1_1-ID

▷ Keep doing this until . . .

$$\Gamma(\emptyset) \cup \Gamma(\Gamma(\emptyset)) \cup \Gamma(\Gamma(\emptyset) \cup \Gamma(\Gamma(\emptyset)))$$

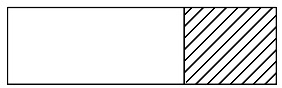

Image of Σ^1_1-ID

▷ until the fixed point:

$\Gamma(F) \subseteq F$ Fixed point

$\boxed{\quad \cdots \quad}$ F

$$F = \{x : (x, x) \in V\} = \text{field}(V)$$

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- $\mathcal{C}$-$\mathrm{ID}_0 \to \mathcal{C}$-$\mathrm{CA}_0$
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- $\mathcal{C}$-$\mathrm{ID}_0 \to \mathcal{C}$-$\mathrm{CA}_0$
- Even Borel determinacy does not imply Δ^1_2-CA.

$\Rightarrow$ Logical equivalence can not be obtained by CA for Σ^0_2-Det.

Even Borel det. doesn't deduce Δ^1_2-CA

(MedSalem, Tanaka)[2007]

($\because$) By β-model reflection and 2nd Imcomp.

- Δ^1_1-Det+Σ^1_2-DC$_0$ $\vdash$ "exist. of c.c. β-model M of Δ^1_1-Det." (by Σ^1_2-DC$_0 \vdash \Sigma^1_4$-RFN^1)
- M $\models \Delta^1_1$-Det $+ \Delta^1_2$-CA$_0$ (By Δ^1_1-Det $\vdash \Delta^1_2$-CA$_0$)
- M$\models$ Δ^1_1-Det $+$ Σ^1_2-DC$_0$ (by Σ^1_2-DC$_0 \leftrightarrow \Sigma^1_\infty$-IND $+ \Delta^1_2$-CA$_0$) $\square$

More details,see M.T[2007],Simpson[2009],VII

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Some Varieties of Σ^1_1-ID_0

- Σ^1_1-ID $\leftrightarrow$ Σ^0_2-Det (1991)
- Σ^1_1-IDTR_0 $\leftrightarrow$ $\Delta((\Sigma^0_2)_2)$-Det (2012)
- $[\Sigma^1_1]^2$-ID_0 $\leftrightarrow$ $(\Sigma^0_2)_2$-Det (2008)

$\vdots$

- $[\Sigma^1_1]^k$-ID_0 $\leftrightarrow$ $(\Sigma^0_2)_k$-Det (2008)
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φ is a $\Delta(\mathcal{C})$ formula if

$$\varphi \leftrightarrow \psi \wedge \neg\varphi \leftrightarrow \eta$$

,where $\psi, \eta \in \mathcal{C}$.

For any $k > 1$, $(\Sigma^0_n)_k = \Sigma^0_n \wedge (\Pi^0_n)_{k-1}$.

e.g. $(\Sigma^0_2)_2 = \Sigma^0_2 \wedge \Pi^0_2$

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Image of Σ^1_1-IDTR$_0$

$\vdots$

$\Gamma^{F^{\prec r}}$ $\quad F^{\prec r} \supset \Gamma^{F^r}(F^r)$
(F^r: fixed point)

$\vdots$

$\Gamma^{F^{r_0}}$ $\quad F^{r_1} \supset \Gamma^{F^{r_1}}(F^{r_1})$
(F^{r_1}: fixed point)

$\Gamma^{\emptyset}$ $\quad F^{r_0} \supset \Gamma^{\emptyset}(F^{r_0})$
(F^{r_0}:fixed point)

where $F^{\prec r} = \cup\{F^{r'} : r' \prec r\}]$

Related Results

- Compare the determinacy strength between Baire space and Cantor Space (Nemoto, Med-Salem, and Tanaka) such as:

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(Det*: determinacy on Cantor sapce).

▷ WKL $\leftrightarrow$ Σ^0_1-Det*

▷ ATR $\leftrightarrow$ Σ^0_1-Det $\leftrightarrow$ Σ^0_2-Det*

▷ $[\Sigma^1_1]^k$-ID $\leftrightarrow$ $(\Sigma^0_2)_k \leftrightarrow$ $(\Sigma^0_2)_{k-1}$-Det*

(for $k \geq 2$)

Related Results

● The limit of determinacy in second order arithmetic: (Montalbán, Shore, 2012)

(i)

(ii)

▷

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(i) Π^1_{m+2}-CA $\rightarrow (\Sigma^0_3)_m$-Det $(m \geq 1)$

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▷ Note that the reversal of (i) does not hold since Δ^1_1-Det $\not\rightarrow \Delta^1_2$-CA

Related Results

- **Set Theory**

▷ Open determinacy for class games for Con(ZFC), (Hamkins)

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