How unprovable is Rabin's decidability theorem?

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What is Rabin's decidability theorem?

Rabin's theorem (1969)

The monadic second order (MSO) theory of the infinite binary tree in the language with two successors, $(\{0,1\}^{<\mathbb{N}}, S_0, S_1)$*, is decidable.*

- Among the most important decidability results in logic.
- Unlike other such results (Presburger, RCF, MSO for $(\mathbb{N}, \leq)$), seems like it might require strong axioms.
- Typical proofs involve a determinacy principle unprovable in Π^1_2-CA_0.

Question:

How much logical strength is needed to prove Rabin's theorem?

Executive summary of the talk

Rabin's theorem

MSO theory of $(\{0,1\}^{<\mathbb{N}}, S_0, S_1)$ *is decidable.*

(By undefinability of truth, it's hard to state this in full in Z_2. But the interesting phenomena appear already for Π^1_3 fragment of MSO.)

Executive summary of the talk

Rabin's theorem

MSO theory of $(\{0,1\}^{<\mathbb{N}}, S_0, S_1)$ *is decidable.*

(By undefinability of truth, it's hard to state this in full in Z_2. But the interesting phenomena appear already for Π^1_3 fragment of MSO.)

Main result:

All forms of Rabin's theorem that can be meaningfully stated in Z_2 are provable in Π^1_3-CA_0 but not in Δ^1_3-CA_0.

Proofs rely on:

- well-known results and techniques from automata theory,
- work on determinacy principles for $\mathrm{Bool}(\mathbf{\Sigma}^0_2)$ games in Z_2 (MedSalem, Nemoto,Tanaka; Heinatsch, Möllerfeld).

What can be said in MSO on $\{0,1\}^{<\mathbb{N}}$?

MSO: $S_0(v,w), S_1(v,w), v \in X, \neg, \vee, \wedge, \exists v, \exists X$ (for X unary!).

MSO can say:

- "v is an ancestor of w":
  every X containing v and closed under S_0, S_1 also contains w".
- A given subset is a path, something happens on all paths etc.
- "All open games in Cantor space are determined" (and more!).
- Can interpret Presburger arithmetic, using finite sets as numbers.

But there is no pairing function, so no chance to get full arithmetic.

Rabin's theorem: proof sketch

- Work with labelled trees: $(\{0,1\}^{<\mathbb{N}}, S_0, S_1, P_{a_1}, \ldots, P_{a_\ell})$ where $\{0,1\}^{<\mathbb{N}} = \bigsqcup_i P_{a_i}$ (vertex in P_{a_i} is "labelled" with letter a_i).
- By induction on MSO sentence φ, show that φ is equivalent on labelled trees to a nondeterministic tree automaton.
- The difficult induction step is for $\neg$ (nondeterminism!).
- This step involves a determinacy principle for parity games.
- It remains to find decision procedure for "given automaton $\mathcal{A}$, does it accept any tree at all?" This is easy.

Tree automata: definition

A nondeterministic tree automaton $\mathcal{A}$ is given by:

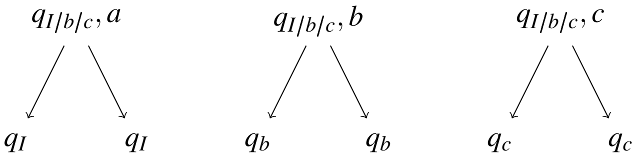
- set of letters $\Sigma = \{a_1, \ldots, a_n\}$ (the alphabet),
- finite set of states Q,
- initial state $q_I \in Q$,
- transition relation $\Delta \subseteq Q \times \Sigma \times Q \times Q$,
- rank function $\mathrm{rk}\colon Q \to \mathbb{N}$.

Idea ("like finite automata, but on infinite trees"):

- Run of $\mathcal{A}$ on tree T labels T with states: vertex $\varnothing$ gets label q_I.
- $\Delta \ni (q, a, q_0, q_1)$ means: if run reaches v in state q and reads a, then it can go to $v0$ in state q_0 and $v1$ in state q_1 simultaneously.
- Run is accepting if on each path, $\liminf$ of ranks of states is even.
- $\mathcal{A}$ accepts T if there is an accepting run on T. (Note: this is Σ^1_2.)

Tree automata: an example

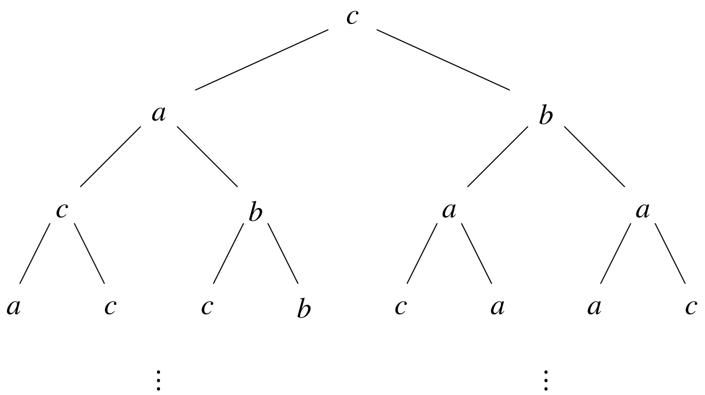
Let $\mathcal{A}$ have alphabet $\{a, b, c\}$, states q_I of rank 2, q_b of rank 1, q_c of rank 0, and transitions:



Then $\mathcal{A}$ accepts exactly a tree T iff on each branch there are either infinitely many c's or only finitely many b's.

Tree automata: an example (cont'd)

$\mathcal{A}$ has alphabet $\{a, b, c\}$, states q_I, q_b, q_c.



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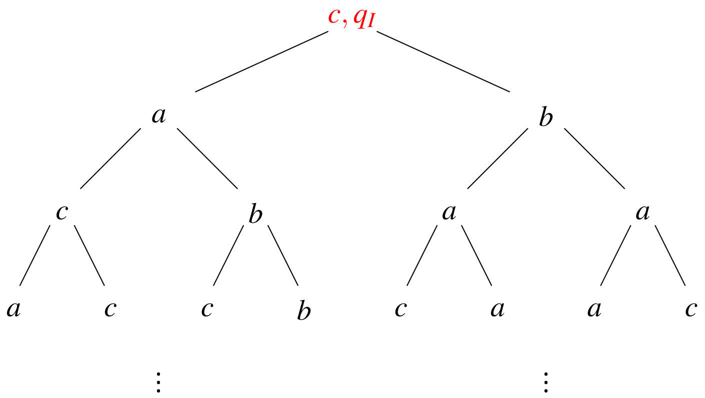

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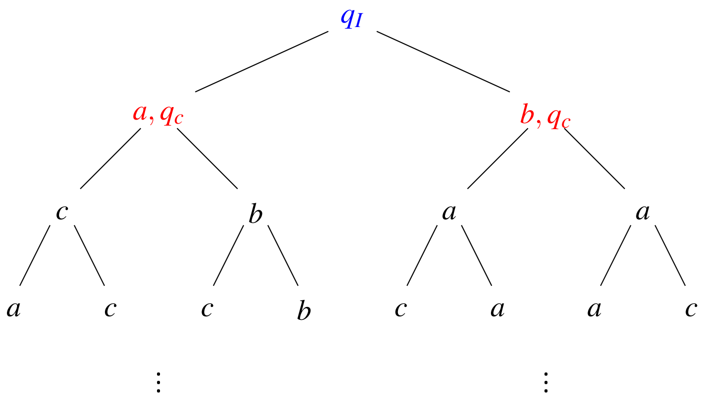

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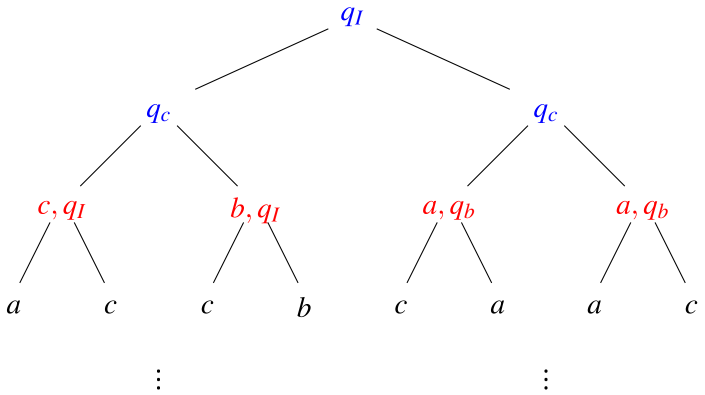

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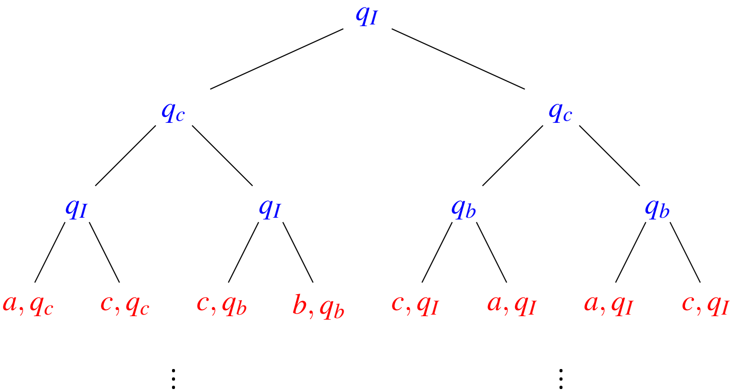


Parity games: definition

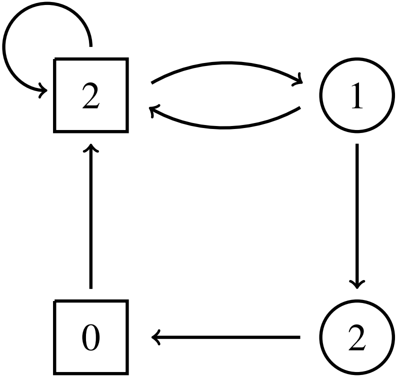
For $k \in \mathbb{N}$, a parity game with ranks up to k is given by:

- finite or countable set $V = V_0 \sqcup V_1$ (the arena, or set of positions),
- initial position $v_0 \in V$,
- edge relation $E \subseteq V^2$,
- rank function $\mathrm{rk}\colon V \to \{0, 1, \ldots, k\}$.

Idea:

- two players: 0 and 1,
- starting in v_0, move to positions $v_1, v_2, \ldots$ along edges,
- player P chooses move from v_i iff $v_i \in V_P$,
- player 0 wins iff $\liminf_{i\to\infty} \mathrm{rk}(v_i)$ is even.

Parity games: an example



Here ◯ is player 0 and □ is player 1. Game starts in upper left.
Player 0 has a winning strategy.

Parity games: determinacy

Observation (in ACA_0, say):

“All parity games are determined”

$$\Updownarrow$$

“All $\mathrm{Bool}(\mathbf{\Sigma}^0_2)$ games are determined”.

(Are the $\mathrm{Bool}(\mathbf{\Sigma}^0_2)$ games in Cantor space or Baire space?
Doesn't matter, cf. MedSalem-Nemoto-Tanaka.)

Important fact:

Parity games enjoy positional (memoryless, forgetful) determinacy:
winning strategy can look at current position ignoring earlier ones!

Rabin's theorem: proof sketch, revisited

- Work with labelled binary trees.
- By induction on MSO sentence φ, show that φ is equivalent to a nondeterministic tree automaton.
- The difficult induction step is for $\neg$.
  (The automata are nondeterministic!)
- This step involves a determinacy principle for parity games.
- It remains to find decision procedure for "given automaton $\mathcal{A}$, does it accept any tree?" This is easy.

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  (The complementation theorem for tree automata).
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Complementation for tree automata

Theorem (Rabin)

For every tree automaton $\mathcal{A}$ there exists a tree automaton $\mathcal{B}$ such that for any tree T, $\mathcal{B}$ accepts T iff $\mathcal{A}$ does not accept T.

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For every tree automaton $\mathcal{A}$ there exists a tree automaton $\mathcal{B}$ such that for any tree T, $\mathcal{B}$ accepts T iff $\mathcal{A}$ does not accept T.

Theorem

Over ACA_0, the above complementation theorem:

(i) *follows from "all parity games are positionally determined",*

(ii) *implies* $\mathrm{Bool}(\boldsymbol{\Sigma}^0_2)$-Det *("all* $\mathrm{Bool}(\boldsymbol{\Sigma}^0_2)$ *games are determined").*

Remark:

The exactly equivalent principle is positional determinacy for a certain class of parity games.

Positional determinacy ⇒ complementation

Proof sketch:

- We formalize a standard proof.
- Main observation: "$\mathcal{A}$ accepts T" is the same as "Player 0 wins in a certain parity game $G_{\mathcal{A},T}$" (Automaton-Pathfinder game).
- By positional determinacy "$\mathcal{A}$ does not accept T" is "Player 1 wins in game $G_{\mathcal{A},T}$ using a positional strategy".
- The latter can be translated into a tree automaton. (Translation is nontrivial and relies on complementation for automata on infinite strings, which is provable in ACA_0.) □

Complementation ⇒ Bool(Σ^0_2)-Det

Proof sketch:

- Given $x \in \mathbb{N}$, games with $\mathrm{Diff}_x(\Sigma^0_2)$ winning condition can be represented by labelled binary trees over fixed alphabet.
- "Game represented by T is not determined" can be written as MSO sentence φ with $4 \pm \epsilon$ quantifier blocks, $\epsilon \in [0, 10]$.
- Complementation applied $\leq 4 + \epsilon$ times transforms φ into $\mathcal{A}_\varphi$.
- Known fact: if automaton accepts any tree, then it accepts a very simple ("regular") tree.
- Easy: game given by regular tree has to be determined.
- So, $\mathcal{A}_\varphi$ rejects all trees! □

Determinacy and comprehension

Theorem (MedSalem-Tanaka)

$\Pi^1_2\text{-}\mathsf{CA}_0 \vdash \mathbf{\Sigma}^0_2\text{-Det} \wedge \forall x\,[\mathrm{Diff}_x(\mathbf{\Sigma}^0_2)\text{-Det} \Rightarrow \mathrm{Diff}_{x+1}(\mathbf{\Sigma}^0_2)\text{-Det}].$

Theorem (Heinatsch-Möllerfeld)

$\{\mathrm{Diff}_n(\mathbf{\Sigma}^0_2)\text{-Det} : n \in \omega\}$ *implies all* Π^1_1 *consequences of* $\Pi^1_2\text{-}\mathsf{CA}_0$.

Corollary (essentially MedSalem-Tanaka)

$\Pi^1_2\text{-}\mathsf{CA}_0 \nvdash \mathrm{Bool}(\mathbf{\Sigma}^0_2)\text{-Det}.$

Theorem

$\Pi^1_2\text{-}\mathsf{CA}_0$ *proves: for every x, if all parity games with ranks up to x are positionally determined, then so are all games with rank up to* $x+1$.

How unprovable is complementation for automata?

Theorem

The complementation theorem for tree automata is:

(i) *provable in* Π^1_2-$\mathsf{CA}_0 + \Pi^1_3$-IND*, and thus also in* Π^1_3-CA_0*,*

(ii) *unprovable in* Π^1_2-CA_0 *and thus also in* Δ^1_3-CA_0*.*

Proof.

Immediate corollary of the determinacy characterization and MedSalem-Tanaka. □

What about the decidability theorem itself?

How unprovable is Rabin's decidability theorem?

Theorem

Over Π^1_2-CA_0*, the statement "the* Π^1_3 *(or* Π^1_4, Π^1_5 *etc.) fragment of the MSO theory of* $(\{0,1\}^{\mathbb{N}}, S_0, S_1)$ *is decidable":*

(i) *follows from "all parity games are positionally determined",*

(ii) *implies* $\mathrm{Bool}(\boldsymbol{\Sigma}^0_2)$-Det.

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Over Π^1_2-CA_0, the statement "the Π^1_3 (or Π^1_4, Π^1_5 etc.) fragment of the MSO theory of $(\{0,1\}^{\mathbb{N}}, S_0, S_1)$ is decidable":

(i) *follows from "all parity games are positionally determined",*

(ii) *implies* $\mathrm{Bool}(\mathbf{\Sigma}^0_2)$-Det.

Proof of (ii):

- Given $x \in \mathbb{N}$, exists Π^1_3 MSO sentence ψ_x expressing "all $\mathrm{Diff}_x(\mathbf{\Sigma}^0_2)$ games are determined".
- Assume e decides the Π^1_3 fragment of MSO.
- Provably in Π^1_2-CA_0, $\forall x\,[e(\psi_x) = 1 \Rightarrow e(\psi_{x+1}) = 1]$.
- By induction, $\forall x\,[e(\psi_x) = 1]$.

Rabin's theorem as a reflection principle

Up to now, we relied on earlier results on determinacy in Z_2.
By analyzing techniques used to prove those results, we can get:

Theorem

For any fixed $n \geq 3$, t.f.a.e. over Π^1_2-CA_0:

1. $\mathrm{Bool}(\boldsymbol{\Sigma}^0_2)$-Det,
2. *positional determinacy of all parity games,*
3. *the complementation theorem for tree automata,*
4. *decidability of the Π^1_n fragment of MSO on $(\{0,1\}^{\mathbb{N}}, S_0, S_1)$,*
5. Π^1_3*-reflection for* Π^1_2-CA_0.

Rabin as reflection: proof ingredients

(o) $\{\mathrm{Diff}_n(\mathbf{\Sigma}^0_2)\text{-Det} : n \in \omega\}$ implies all Π^1_1 theorems of Π^1_2-CA_0. (Heinatsch-Möllerfeld).

(i) (o) can be improved (by careful analysis of role of Axiom β): $\{\mathrm{Diff}_n(\mathbf{\Sigma}^0_2)\text{-Det} : n \in \omega\}$ axiomatizes Π^1_3 theorems of Π^1_2-CA_0.

(ii) (i) can be formalized in reasonably weak theory (apparently in PRA, but even Π^1_2-CA_0 would still be ok).

(iii) To get from (ii), we need an argument about β_2 models.

Executive summary, once more

Rabin's theorem

MSO theory of $(\{0,1\}^{<\mathbb{N}}, S_0, S_1)$ *is decidable.*

Main result:

All forms of Rabin's theorem that can be meaningfully stated in Z_2 are provable in Π^1_3-CA_0 but not in Δ^1_3-CA_0.

Proofs rely on:

- well-known results and techniques from automata theory,
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Further work

- Do the equivalences we prove in Π^1_2-CA_0 hold in ACA_0?
- Is there a more general connection between determinacy and Π^1_3-reflection?
- What is the exact logical strength needed to prove decidability of the MSO theory of $(\mathbb{N}, \leq)$?