Some principles weaker than Markov's principle

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Constructive Mathematics (Early 20th Century –)

- Constructive mathematics is distinguished from its traditional counterpart, classical mathematics, by the strict interpretation of the phrase "there exists" as "we can construct".*

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- In order to work constructively, we need to re-interpret not only the existential quantifier but all the logical connectives and quantifiers as instructions on how to construct a proof of the statement involving these logical expressions (BHK-interpretation).
- Heyting (1930's -) and Kolmogorov (1920's -) tried to formalize constructive mathematics and introduced **intuitionistic logic**.

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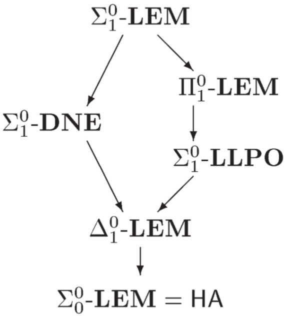
Heyting Arithmetic HA

- As language, HA has variables (for natural numbers), 0, successor S, function constants for all primitive recursive functions and a binary predicate constant $=$.
- HA is based on intuitionistic first order predicate logic and in addition contains
  - the defining axioms for the primitive recursive function constants,
  - the equality axioms,
  - IND: $A(0) \wedge \forall x\,(A(x) \to A(Sx)) \to \forall x A(x)$.

Hierarchy of Logical Principles over HA (Akama, Berardi, Hayashi and Kohlenbach, 2004)



- Γ-LEM: $A \vee \neg A$, where $A \in \Gamma$ ($\Gamma \in \{\Sigma^0_0, \Sigma^0_1, \Pi^0_1\}$).
- Σ^0_1-LLPO: $\neg(A \wedge B) \rightarrow (\neg A \vee \neg B)$, where $A, B \in \Sigma^0_1$.
- Σ^0_1-DNE: $\neg\neg A \rightarrow A$, where $A \in \Sigma^0_1$.
- Δ^0_1-LEM: $(A \leftrightarrow B) \rightarrow (A \vee \neg A)$, where $A \in \Sigma^0_1$, $B \in \Pi^0_1$.

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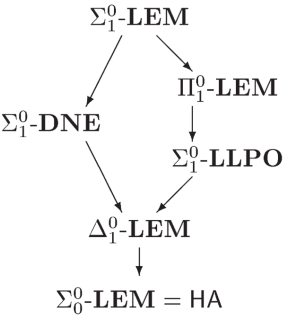

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- Σ^0_1-DNE $\equiv$ MP: $\neg\neg A \to A$, where $A \in \Sigma^0_1$.
- Δ^0_1-LEM: $(A \leftrightarrow B) \to (A \vee \neg A)$, where $A \in \Sigma^0_1$, $B \in \Pi^0_1$.

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Elementary analysis EL is a conservative extension of HA, which is served as base theory formalizing (Bishop-style) constructive mathematics.

- As language, EL has two-sorted variables (for numbers and functions), abstraction operators $\lambda x.$(only for numbers), a recursor R in addition to that for HA.
- Axioms and rules of EL contain
  - λ-CON: $(\lambda x.t)t' = t[t'/x]$
  - REC: $Rt\varphi 0 = 0$ and $Rt\varphi(St') = \varphi(Rt\varphi t', t')$
  - QF-AC$_{0,0}$: $\forall x \exists y A_{qf}(x, y) \to \exists f \forall x A_{qf}(x, fx)$
  - IND: $A(0) \wedge \forall x\, (A(x) \to A(Sx)) \to \forall x A(x)$
- EL$_0$ is a fragment of EL where IND is replaced by QF-IND.

	Intuitionistic Logic	Classical Logic
Non-sorted	HA	PA
Two-sorted	EL	RCA
	EL_0	RCA_0

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Non-sorted	HA	PA
Two-sorted	EL	RCA
	EL_0	RCA_0

- RCA_0 is the most popular base system of reverse mathematics, which consists of
  - basic axioms BA of arithmetic based on **classical** logic,
  - Σ^0_1 induction scheme Σ^0_1-IND,
  - Δ^0_1 comprehension scheme Δ^0_1-CA:

$$\forall \alpha, \beta \left(\begin{array}{l} \forall y \big(\exists x(\alpha(y,x) = 0) \leftrightarrow \neg \exists x(\beta(y,x) = 0)\big) \\ \to \exists \gamma \forall y \big(\gamma(y) = 0 \leftrightarrow \exists x(\alpha(y,x) = 0)\big) \end{array} \right).$$

- RCA consists of BA, IND and Δ^0_1-CA.

Proposition

- EL_0 (containing only $\mathrm{QF\text{-}IND}$) $\vdash \Sigma^0_1\text{-}\mathrm{IND}$.
- $\mathsf{EL}_0 + \mathrm{LEM}(A \vee \neg A) \vdash \Delta^0_1\text{-}\mathrm{CA}$.

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In fact, $\Delta^0_1\text{-}\mathrm{CA}$ is intuitionistically derived from $\mathrm{QF\text{-}AC}_{0,0}$ and Markov's principle MP:

$$\forall\alpha\big(\neg\neg\exists x(\alpha(x) = 0) \rightarrow \exists x(\alpha(x) = 0)\big).$$

- Note that $\Delta^0_1\text{-}\mathrm{CA}$ is equivalent to $\Delta^0_1\text{-}\mathrm{LEM}$ over $\mathsf{EL}_0 + \mathrm{AC}$.

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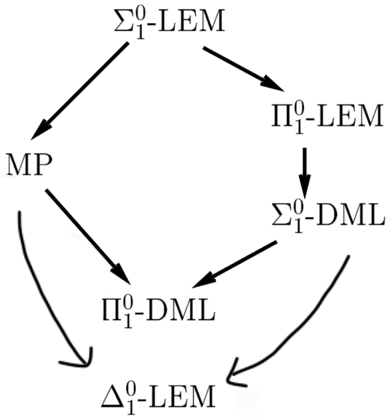
- Note that $\Delta^0_1\text{-}\mathrm{CA}$ is equivalent to $\Delta^0_1\text{-}\mathrm{LEM}$ over $\mathsf{EL}_0 + \mathrm{AC}$.
- Inspecting the proofs in [Akama et al. 2004] reveals that there is also a corresponding hierarchy over EL or EL_0.
- In particular, $\Delta^0_1\text{-}\mathrm{LEM}$ is derived from either MP or $\Sigma^0_1\text{-}\mathrm{DML}$.

Proposition. (Ishihara 1993)

1. $\mathsf{EL}_0 \vdash \mathrm{MP} \to \Pi^0_1\text{-DML}$.
2. $\mathsf{EL}_0 \vdash \Sigma^0_1\text{-DML} \to \Pi^0_1\text{-DML}$.

Note that Π^0_1-DML is denoted as $\mathrm{MP}^\vee$ in the literature.

Situation



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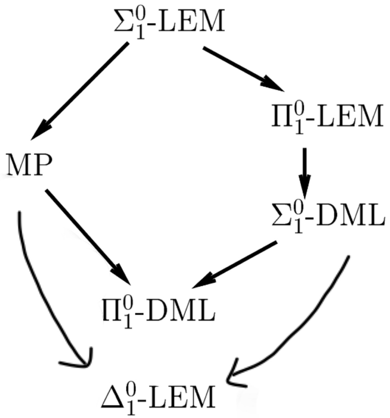

Question.

How is the relationship between Π^0_1-DML and Δ^0_1-LEM?

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Note that Δ^0_1-LEM in [Akama et al. 2004] has been defined in the sense of (a).

$\Rightarrow$ We consider the fragments of LEM with respect to Δ_i ($i \in \{a, b, c, ab\}$).

Definition.

$$\Delta_{\mathrm{i}}\text{-LEM} :\equiv \forall\alpha \left(\begin{array}{l} \alpha \in \Delta_{\mathrm{i}} \to \\ \exists x \alpha(x) = 0 \vee \neg\exists x \alpha(x) = 0 \end{array} \right).$$

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Remark.

$$\begin{array}{ccccc} & & \Delta_{\mathrm{ab}} & & \\ & \swarrow & & \searrow & \\ \Delta_{\mathrm{a}} & & & & \Delta_{\mathrm{b}} \\ & \searrow & & \swarrow & \\ & & \Delta_{\mathrm{c}} & & \end{array}$$

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The following are pairwise equivalent over EL (even over EL_0).

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$\Rightarrow$ How is the converse direction?

Definition.

$\Delta[\mathrm{a} \to \mathrm{b}] :\equiv \forall\alpha(\alpha \in \Delta_{\mathrm{a}} \to \alpha \in \Delta_{\mathrm{b}})$.

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Lemma.

Δ_{a}-LEM implies $\Delta[\mathrm{a} \to \mathrm{b}]$ over EL_0.

Proof. We reason in EL_0. Let $\alpha \in \Delta_{\mathrm{a}}$.
Then $\exists x \alpha(x) = 0 \vee \neg \exists x \alpha(x) = 0$ holds by Δ_{a}-LEM.
In the case of $\exists x \alpha(x) = 0$, take β as $\beta \equiv 1$.
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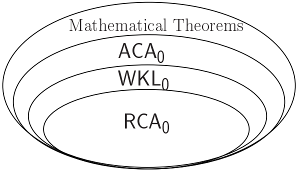
Open Problem. Does Δ_{ab}-LEM imply $\Delta[\mathrm{a} \to \mathrm{b}]$ over EL?

Remark. Reverse Mathathematical Hierarchy vs Hierarchy of Logical Principles

- Reverse Mathematics Phenomenon:



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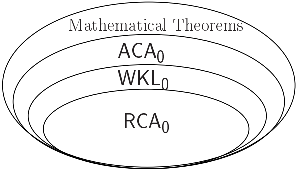

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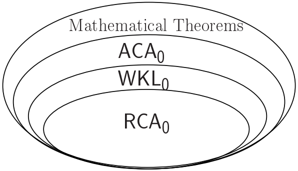

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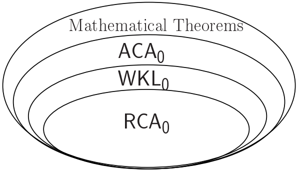

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- However, the corresponding system to Π^0_1-DML or Δ_{i}-LEM is still missing.

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Thank you for your attention!