Axiom schema of Markov’s principle preserves disjunction and existence properties

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Computability Theory and Foundations of Mathematics 2015
September 7, 2015 (Tokyo, Japan)
“Hallmarks” of constructivity of intuitionistic logic \( \mathbf{H}_* \):

**Fact**

\( \mathbf{H}_* \) has the **disjunction property** (DP);
for every \( A \lor B \): \( \mathbf{H}_* \vdash A \lor B \Rightarrow \mathbf{H}_* \vdash A \lor B \).

\( \mathbf{H}_* \) has the **existence property** (EP);
for every \( \exists x A(x) \): \( \mathbf{H}_* \vdash \exists x A(x) \Rightarrow \exists v \quad \text{such that} \quad \mathbf{H}_* \vdash A(v) \).

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N.B. \( A(v) \) should be taken as a formula congruent to \( A \) free from collision of variables.
### Introduction: Disjunction and Existence Properties

“Hallmarks” of constructivity of intuitionistic logic $H_*:

### Fact

**$H_*$ has the disjunction property (DP):**

For every $A \lor B$: $H_* \vdash A \lor B \Rightarrow H_* \vdash A$ or $H_* \vdash B$.

**$H_*$ has the existence property (EP):**

For every $\exists x A(x)$: $H_* \vdash \exists x A(x) \Rightarrow$ there exists a $v$ such that $H_* \vdash A(v)$.

**$H_* + A$:** the logic obtained from $H_*$ by adding the axiom schema $A$.

There are schememata $A$ such that $H_* + A$ enjoys both of DP and EP.

We are interested in such schemata (i.e., $H_* + A$ still enjoys DP and EP) in the setting of Intermediate Predicate Logics, particularly in those schemata related to constructive theories.

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N.B. $A(v)$ should be taken as a formula congruent to $A$ free from collision of variables.
Markov’s Principle and Limited Principle of Omniscience

In the setting of intermediate Predicate Logics, we consider:

**Axiom schema of Markov’s principle:**

\[
MP : \ \forall x (A(x) \lor \neg A(x)) \land \neg \neg \exists x A(x) \rightarrow \exists x A(x).
\]

**Axiom schema of the limited principle of omniscience:**

\[
LPO : \ \forall x (A(x) \lor \neg A(x)) \rightarrow \exists x A(x) \lor \neg \exists x A(x),
\]

Both principles enlarge the concept of constructivity, particularly the concept of \(\exists\) from that of intuitionistic logic \(H_*\).

However, still we have:

**Theorem**

\(H_* + MP\) and \(H_* + LPO\) enjoy DP and EP.
That is, MP and LPO preserve DP and EP.
Harrop-DP and Harrop-EP

**Definition**
A formula is said to be a Harrop-formula (H-formula) if every strictly positive subformula is neither of the form $A \lor B$ nor $\exists x A(x)$.

**Theorem (Harrop)**

$H_\ast$ has the $H$(arrop)-DP and the $H$(arrop)-EP, i.e., for any H-formula $H$,

$H_\ast \vdash H \rightarrow A \lor B \Rightarrow H_\ast \vdash H \rightarrow A$ or $H_\ast \vdash H \rightarrow B$,

$H_\ast \vdash H \rightarrow \exists x A(x) \Rightarrow H_\ast \vdash H \rightarrow A(v)$ for some $v$.

**Theorem**

$H_\ast + MP$ and $H_\ast + LPO$ enjoy $H$-DP and $H$-EP. That is, MP and LPO preserve $H$-DP and $H$-EP.
**Definition**

- **$M_1$, $M_2$:** Kripke frames with the least elements $0_1$ and $0_2$, resp., such that the domains at $0_1$ and $0_2$ coincide with $V (= D_1(0_1) = D_2(0_2))$.

  A Kripke frame $M$ is said to be the **pointed join frame** of $M_1$ and $M_2$, if $M = \{(0, V)\} \uparrow M_1 \oplus M_2$ with a fresh least element $0$.

- **$(M_1, \models_1), (M_2, \models_2)$**: Kripke models with $V = D_1(0_1) = D_2(0_2)$.

  A Kripke model $(M, \models)$ is said to be a **pointed join model** of $(M_1, \models_1)$ and $(M_2, \models_2)$, if $M$ is the pointed join frame of $M_1$ and $M_2$, and the restrictions of $\models$ to $M_1$ and $M_2$ are $\models_1$ and $\models_2$, resp.
Axiomatic Truth and its Preservation

Definition
A formula $A$ is said to be axiomatic truth in a Kripke model $(M, \models)$, if universal closures of all of substitution instances of $A$ are true in $(M, \models)$.

Lemma
If $A$ preserves its axiomatic truth in the construction of pointed join models, i.e., satisfies the following:

- If $A$ is axiomatic truth in Kripke models $(M_1, \models_1)$ and $(M_2, \models_2)$ with $V = D_1(0_1) = D_2(0_2)$, then $A$ is still axiomatic truth in any pointed join model of $(M_1, \models_1)$ and $(M_2, \models_2)$,

then $H_\ast + A$ preserves H-DP and H-EP.
Axiomatic Truth and its Preservation

**Definition**

A formula $A$ is said to be **axiomatically true** in a Kripke model $(M, \models)$, if universal closures of all of substitution instances of $A$ are true in $(M, \models)$.

**Lemma**

If $A$ preserves its axiomatic truth in the construction of pointed join models, i.e., satisfies the following:

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then $H_* + A$ preserves H-DP and H-EP.

**Theorem**

*MP and LPO have this property.*

Hence, $MP$ and $LPO$ preserve H-DP and H-EP.
**Definition**

A formula is said to be a **weak Harrop-formula (wH-formula)** if every strictly positive subformula is not of the form $\exists x A(x)$.

**Theorem (Prawitz, Doorman)**

$H_*$ has the **Prawitz-Doorman EP**, i.e.,

for any wH-formula $H$,

$H_* \vdash H \rightarrow \exists x A(x) \Rightarrow$

there exist finitely many $v_1, \ldots, v_n$ in the vocabulary of $H \rightarrow \exists x A(x)$ such that $H_* \vdash H \rightarrow A(v_1) \lor \cdots \lor A(v_n)$.

Prawitz proved EP of $H_*$ by showing DP and the Prawitz-Doorman EP.
Another Phenomenon: Prawitz-Doorman EP

Definition

A formula is said to be a weak Harrop-formula (wH-formula) if every strictly positive subformula is not of the form $\exists x A(x)$.

Theorem (Prawitz, Doorman)

$H_\ast$ has the Prawitz-Doorman EP, i.e.,
for any wH-formula $H$,
$H_\ast \vdash H \to \exists x A(x) \Rightarrow$
there exist finitely many $v_1, \ldots, v_n$ in the vocabulary of $H \to \exists x A(x)$
such that $H_\ast \vdash H \to A(v_1) \lor \cdots \lor A(v_n)$.

Prawitz proved EP of $H_\ast$ by showing DP and the Prawitz-Doorman EP.

Proposition

$H_\ast + MP$ and $H_\ast + LPO$ fail to have the Prawitz-Doorman EP.
That is, $MP$ and $LPO$ do not preserve the Prawitz-Doorman EP.
In this talk, we considered preservation of DP and EP by two schemata $MP$ and $LPO$ in the setting of intermediate predicate logics.

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<tbody>
<tr>
<td>$H_*$</td>
<td>YES</td>
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<tr>
<td>$H_* + MP, H_* + LPO$</td>
<td>YES</td>
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Concluding Remarks (1)

In this talk, we considered preservation of DP and EP by two schemata $MP$ and $LPO$ in the setting of intermediate predicate logics.

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<tr>
<td>$H_* + WLPO, H_* + LLPO$</td>
<td>?</td>
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<tr>
<td>$H_* + CD$</td>
<td>YES</td>
<td>NO</td>
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<tr>
<td>$H_* + WEM$</td>
<td>NO</td>
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$WLPO$: $\forall x(p(x) \lor \lnot p(x)) \rightarrow \lnot \exists xp(x) \lor \lnot \lnot \exists xp(x)$,

$LLPO$: $\{\forall x(p(x) \lor \lnot p(x)) \land \forall x(q(x) \lor \lnot q(x)) \land \lnot (\exists xp(x) \land \exists xq(x))\} \rightarrow \lnot \exists xp(x) \lor \lnot \exists xq(x)$,

$CD$: $\forall x(p(x) \lor q) \rightarrow \forall xp(x) \lor q$, ($x$ is not free in $q$)

$WEM$: $\lnot p \lor \lnot \lnot p$,
### Relations?

In intermediate predicate logics: \( \text{DP} \Rightarrow \text{EP?} \quad \text{EP} \Rightarrow \text{DP?} \)

- (Nakamura 1983) There exists an intermediate logic having DP but lacking EP. I.e., \( \text{DP} \nRightarrow \text{EP} \).
- \( \text{EP} \Rightarrow \text{DP?} \) in intermediate logics
In intermediate predicate logics, EP and DP are independent. I.e.,

- (Nakamura 1983) There exists an intermediate logic having DP but lacking EP. I.e., DP \( \not\Rightarrow \) EP.
- (S. 2013-15) There exists an intermediate logic having EP but lacking DP. I.e., EP \( \not\Rightarrow \) DP.

**Theorem (S.2013-15)**

If \( L \) is closed under the rule:

\[
A \lor (p(x) \rightarrow p(y)) \quad \frac{A}{A} \quad (ZR)
\]

where \( x, y \) and \( p \) are distinct and do not occur in \( A \).

Then, EP of \( L \) implies DP of \( L \).
Do $H^*_+WLPO$ and $H^*_+LLPO$ have H-DP and/or H-EP?

H-DP $\Leftrightarrow$ DP? H-EP $\Leftrightarrow$ EP?

This problem is known as Ono’s problem P54.

Remark: In intermediate propositional logic, we have: H-DP $\Leftrightarrow$ DP.

There must be waiting us other axiom schemata arising from constructive theories which are interesting from the viewpoint of intermediate logics!

There must be waiting us other phenomena in intermediate logics which are interesting from the viewpoint of constructive theories!


