The Complexity of Isomorphism

An important equivalence relation across all of mathematics is the relation of *isomorphism*.

As logicians it is natural to pose the question: How complex is the isomorphism relation in comparison with other equivalence relations?

The answer to this question is sensitive to the class of structures under consideration; we consider five cases:

- Countable structures
- Computable structures
- Hyp (Hyperarithmetic $= \Delta^1_1$) structures
- Uncountable structures
- Finite structures

The Complexity of Isomorphism

In all five cases, isomorphism is Σ^1_1; more precisely:

$\mathcal{A}$, $\mathcal{B}$ are isomorphic iff

$\exists F : A \to B(F$ is a structure-preserving bijection)

- Countable structures can be coded by reals; then isomorphism becomes a Σ^1_1 relation on reals
- Computable structures can be coded by natural numbers; then isomorphism becomes a Σ^1_1 relation on ω
- Hyp structures can be coded by Hyp reals; then isomorphism becomes the restriction of a Σ^1_1 relation to the Hyp reals
- Structures of size κ can be coded by subsets of κ; then isomorphism becomes a Σ^1_1 relation on generalised Cantor space 2^κ
- Finite structures can be coded by finite strings; then isomorphism becomes a Σ^1_1 or NP relation on finite strings

The Complexity of Isomorphism

The Basic Question. Is isomorphism Σ^1_1 complete?

This translates into five questions:

- Suppose that $R(x, y)$ is a Σ^1_1 equivalence relation on reals. Is there a Borel function f from reals to countable structures such that $R(x, y)$ iff $f(x), f(y)$ are isomorphic?
- Suppose that $R(m, n)$ is a Σ^1_1 equivalence relation on ω. Is there a Hyp function f from ω to computable structures such that $R(m, n)$ iff $f(m), f(n)$ are isomorphic?
- Suppose that $R(x, y)$ is a Σ^1_1 equivalence relation on reals. Is there a Hyp function f from reals to countable structures such that for Hyp x, y, $R(x, y)$ iff $f(x), f(y)$ are isomorphic?

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- Suppose that $R(x, y)$ is a Σ^1_1 equivalence relation on subsets of κ. Is there a κ-Borel function f from subsets of κ to structures on κ such that $R(x, y)$ iff $f(x), f(y)$ are isomorphic?
- Suppose that $R(s, t)$ is an NP equivalence relation on finite strings. Is there a Polytime function f from finite strings to finite structures such that $R(s, t)$ iff $f(s), f(t)$ are isomorphic?

These are questions of descriptive set theory (countable case), computable structure theory (computable case), Hyp theory (Hyp case), generalised descriptive set theory (uncountable case) and complexity theory (finite case). And they have interesting answers.

The Complexity of Isomorphism for Countable Structures

Again, in the countable case, we are asking:

- Suppose that $R(x, y)$ is a Σ^1_1 equivalence relation on reals. Is there a Borel function f from reals to countable structures such that $R(x, y)$ iff $f(x), f(y)$ are isomorphic?

In the above we say that R is *Borel-reducible* to isomorphism and f is a *Borel reduction* witnessing this.

Dana Scott answered this negatively long ago:

The Complexity of Isomorphism for Countable Structures

Theorem

There are Σ^1_1 equivalence relations on reals which are not Borel-reducible to Isomorphism $\simeq$.

Proof. Let X be a set of reals which is Σ^1_1 but not Borel.
Define: $x \ E_X \ y$ iff $x, y \in X$ or $x = y$

Then E_X is Σ^1_1 and X is a non-Borel equivalence class of E_X.
But:

Theorem

(Scott) The equivalence classes of $\simeq$ are Borel, i.e., if A is a countable structure then the set $[A]_\simeq$ of codes for structures B which are isomorphic to A forms a Borel set.

It follows that E_X cannot Borel-reduce to $\simeq$ $\square$

The Complexity of Isomorphism for Countable Structures

I should mention that this is far from the end of the story in the countable case: Hjorth developed a deep theory of *turbulence* which explains when equivalence relations induced by group actions are Borel-reducible to isomorphism and this is still an active area of research. Moreover, isomorphism on specific Borel classes of structures yields equivalence relations of different complexities and this continues to be heavily investigated.

The Complexity of Isomorphism for Computable Structures

The picture is very different in the computable setting. Recall that we are now asking:

- Suppose that $R(m, n)$ is a Σ^1_1 equivalence relation on ω. Is there a Hyp function f from ω to computable structures such that $R(m, n)$ iff $f(m), f(n)$ are isomorphic?

In the above we say that R is *Hyp-reducible* to isomorphism and f is a *Hyp reduction* witnessing this.

(By a "Hyp function" I mean a function whose graph is Δ^1_1 or equivalently Hyperarithmetic.)

The Complexity of Isomorphism for Computable Structures

Theorem

(Fokina-Harizanov-Knight-McCoy-Montalban-me) Every Σ^1_1 equivalence relation on ω is Hyp-reducible to $\simeq$ on the computable structures. (i.e., $\simeq$ for computable structures is Σ^1_1-complete).

Proof Sketch: Let E be a Σ^1_1 equivalence relation on ω and fix a computable $f : \omega^2 \to$ *Computable Trees* such that $\sim mEn$ iff $f(m, n)$ is wellfounded. This is possible as $\sim E$ is Π^1_1 and any Π^1_1 set is effectively reducible to the set of wellfounded computable trees.

The Complexity of Isomorphism for Computable Structures

Now associate to pairs m, n better computable trees $T(m, n)$ so that:

1. $T(m, n)$ is isomorphic to $T(n, m)$
2. mEn implies that $T(m, n)$ is isomorphic to the "canonical" illfounded computable tree
3. $\sim mEn$ implies that $T(m, n)$ is isomorphic to the "canonical" computable tree of rank α, where α is independent of the choice of m, n in $[m]_E, [n]_E$, respectively.

In 3 we first get a tree $T'(m, n)$ of rank α by considering all finite sequences $(a_0, \ldots, a_l)$ beginning with m and ending with n, interlacing the trees $f(a_i, a_{i+1})$ for $i < l$ and finally putting together the resulting trees for all such finite sequences $(a_0, \ldots, a_l)$ into one tree. To get the "canonical" computable tree $T(m, n)$ of rank α we interlace $T'(m, n)$ with the "canonical" illfounded computable tree.

Now to each n associate the tree T_n gotten by gluing together the $T(n,i)$, $i \in \omega$.

If mEn then T_m is isomorphic to T_n as they are obtained by gluing together isomorphic trees. This is because the isomorphism-types of the trees $T(m,n)$ are independent of the choice of m,n in $[m]_E, [n]_E$, respectively.

And if $\sim mEn$ then T_m, T_n are not isomorphic as they are obtained by gluing together trees which on some component are non-isomorphic.

So we have mEn iff T_m, T_n are isomorphic, giving the desired Hyp (indeed computable) reduction. $\square$

The Complexity of Isomorphism for Computable Structures

So isomorphism of computable trees is Σ^1_1 complete.
It can be shown that this relation Hyp-reduces to isomorphism on each of the following Hyp classes:

1. Computable graphs
2. Computable torsion-free Abelian groups
3. Computable Abelian p-groups for a fixed prime p
4. Computable Boolean Algebras
5. Computable linear orders
6. Computable fields

and therefore these are also Σ^1_1 complete. 2 and 3 are a bit surprising, as for countable structures, the Σ^1_1 completeness of isomorphism for torsion-free Abelian groups is a major open problem and for Abelian p-groups is provably false!

The Complexity of Isomorphism for Computable Structures

In fact it is still not known if there is any isomorphism relation on a Hyp class of computable structures which is neither Hyp nor Σ^1_1 complete! Montalban has shown that this question is connected with Vaught's Conjecture.

However we do have:

Theorem

(Fokina-me) Every Σ^1_1 equivalence relation on ω is Hyp bi-reducible with bi-embeddability on a Hyp class of computable structures.

This is a Hyp analogue of my result with Motto Ros that any Σ^1_1 equivalence relation on reals is Borel bi-reducible with bi-embeddability on a Borel class of countable structures.

The Complexity of Isomorphism

Now is isomorphism Σ^1_1 complete on classes of structures which lie strictly between Computable and Countable?

Let's use the L-hierarchy to define such classes of structures. For α a countable ordinal in L and $n < \omega$ define:

$\mathcal{C}(\alpha, n)$ = all countable structures coded by reals which are Δ_n definable over L_α with parameters

So Computable $= \mathcal{C}(\omega, 1)$ and Countable $=$ the union of all of the $\mathcal{C}(\alpha, n)$'s. It turns out that using Scott's work and a bit of fine structure theory one can reduce the analysis of all of these cases to just two cases:

$\mathcal{C}(\omega, 1) =$ Computable
$\mathcal{C}(\omega_1^{ck}, 1) =$ Hyperarithmetic.

The Complexity of Isomorphism for Hyp Structures

The Computable case has already been handled, so we are now asking:

- Suppose that $R(x, y)$ is a Σ^1_1 equivalence relation on reals. Is there a Hyp function f from reals to countable structures such that for Hyp x, y, $R(x, y)$ iff $f(x), f(y)$ are isomorphic?

The method used in the Computable case does not seem to work for the Hyp case: There is a Hyp enumeration of the computable reals but no Hyp enumeration of all Hyp reals.

The Scott method does not seem to work either: If $\mathcal{A}$ has a Hyp code there need not be a Borel set $\mathbb{B}$ with Hyp code such that $[A]_{\simeq} \cap Hyp = \mathbb{B} \cap Hyp$.

The solution comes from a deeper look at descriptive set theory and infinitary logic.

The Complexity of Isomorphism for Hyp Structures

For $x \subseteq \omega$ and $n \in \omega$ define $(x)_n = \{m \mid \langle m, n \rangle \in x\}$, where $\langle .,. \rangle$ is a computable pairing function on ω.

The equivalence relation E_1 is defined by:

$$x \; E_1 y \text{ iff } (x)_n = (y)_n \text{ for large enough } n.$$

E_1 is a Hyp equivalence relation. A classic result of Kechris-Louveau is the following:

Theorem

E_1 is not Borel-reducible to isomorphism on countable structures.

The essential difficulty in applying the proof of this to the Hyp case is that two Hyp structures can be isomorphic without being Hyp isomorphic. However this does not happen for Hyp structures of low (computable) Scott rank and we at least have:

The Complexity of Isomorphism for Hyp Structures

Theorem

There is no Hyp f such that for Hyp reals x, y, xE_1y iff $f(x)$ is isomorphic to $f(y)$, and for each Hyp x, $f(x)$ is a structure of low Scott rank.

So to complete the argument that isomorphism is not Σ^1_1 complete on Hyp, we need a method for converting arbitrary structures to structures of low Scott rank.

Let $\equiv_\alpha$ denote elementary equivalence for sentences of $L_{\omega_1\omega}$ of rank less than α.

Theorem

For each computable limit ordinal α there is a Hyp reduction of the equivalence relation $\equiv_\alpha$ on countable structures to isomorphism on countable structures of Scott rank at most α.

The Complexity of Isomorphism for Hyp Structures

Putting the above together we get:

Theorem

There is no Hyp function f such that for Hyp reals x, y, xE_1y iff $f(x)$ is isomorphic to $f(y)$. In particular, isomorphism is not Hyp-complete (and hence not Σ^1_1 complete) on Hyp.

Question. Suppose that E is a Σ^1_1 equivalence relation and E_1 is not Hyp-reducible to E on Hyp. Then is E Hyp-reducible to isomorphism on Hyp?

The answer to this question is likely to be "No", as probably there are "effective orbit equivalence relations" more complex than isomorphism on Hyp to which E_1 cannot be Hyp-reduced.

The Complexity of Isomorphism for Uncountable Structures

Now we turn to uncountable structures. For simplicity let's focus on structures of size κ where κ is the successor of a regular cardinal. Then we get a situation which bears cosiderable resemblance to the computable case.

Theorem

(Hyttinen-Kulikov-me) Assume $V = L$ and let κ be the successor of a regular cardinal. Then all Σ^1_1 equivalence relations are κ-Borel reducible to isomorphism.

I give a hint of the proof. Write $\kappa = \lambda^+$ where λ is regular, let $\mathcal{Q}$ be a λ-saturated dense linear order without endpoints and let $\mathcal{Q}_0$ be $\mathcal{Q}$ together with a least point. For any subset S of κ let $\mathcal{L}(S)$ be obtained from the natural order on κ by replacing α by $\mathcal{Q}_0$ if α is a limit ordinal in S and by $\mathcal{Q}$ otherwise.

Fact. $\mathcal{L}(S)$ is isomorphic to $\mathcal{L}(T)$ iff $S \triangle T$ is nonstationary in κ.

The Complexity of Isomorphism for Uncountable Structures

Now the key Lemma is that in L, any Σ_1^1 set X is κ-Borel reducible to the collection (ideal) of nonstationary sets in the sense that there is a κ-Borel function f such that $x \in X$ iff $f(x)$ is nonstationary. One strengthens this to show that in fact any Σ_1^1 equivalence relation is κ-Borel reducible to equality modulo a nonstationary set and therefore by the above *Fact* to isomorphism of dense linear orders.

Question. Is it consistent that isomorphism on structures of size ω_1 is *not* Σ_1^1-complete (and CH holds)?

Isomorphism on structures of size ω_1 is Σ_1^1 complete as a set, so the real question is whether it is Σ_1^1 complete as an equivalence relation (under unary ω_1-Borel reductions).

The Complexity of Isomorphism for Uncountable Structures

Fokina-Knight-R.Miller and I show that one also gets the Σ^1_1 completeness of isomorphism on the structures of size ω_1 which are ω_1-*computable* (assuming $V = L$). The proof combines the Σ^1_1 completeness arguments for the Computable and Uncountable cases.

The Complexity of Isomorphism for Finite Structures

For finite structures our basic question becomes:

- Suppose that $R(s,t)$ is an NP equivalence relation on finite strings. Is there a Polytime function f from finite strings to finite structures such that $R(s,t)$ iff $f(s), f(t)$ are isomorphic?

This is linked with open questions in computational complexity theory:

The Complexity of Isomorphism for Finite Structures

Proposition

(Buss-Chen-Flum-Müller-me) Assume that the Polytime Hierarchy does not collapse. Then not every NP equivalence relation Polytime-reduces to isomorphism.

Proof. SAT can be turned into an NP equivalence relation:

xEy iff $x = y$ or $x, y \in$ SAT.

Then a Polytime-reduction of E to graph isomorphism (which is Polytime-maximal among isomorphism relations) would imply that graph isomorphism is NP-complete.
It is known that the latter implies that the Polytime hierarchy collapses. □

The Complexity of Isomorphism for Finite Structures

In all of the cases of infinite structures we can at least produce Σ_1^1 equivalence relations which are complete; this is not known in the finite case:

Question. Is there an *NP* equivalence relation which is complete for *NP* equivalence relations under Polytime reductions?

The Complexity of Isomorphism: Summary

To summarise: Isomorphism is Σ^1_1 complete for computable structures and, if $V = L$, for structures of size ω_1. It is not for countable structures or for Hyp structures. It is currently not known if it can fail to be Σ^1_1 complete for finite structures or for structures of size ω_1 if $V \neq L$.

Interesting work in the case of uncountable structures remains to be done and of course it will be interesting to look deeper at the different possible complexities of isomorphism restricted to special classes of structures in each of these five cases.

Thanks for listening, and I would like to wish Professor Tanaka a very Happy 60th Birthday!