A reflection principle as a reverse mathematical fixed point

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CTFM 2015
in honor of Professor Kazuyuki Tanaka's 60th birthday

(16. September 2015 (12:33 JST) version)

11. September 2015, 於 ディジタル多目的ホール (東京工業大学)

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田中一之 先生 (Professor Kazuyuki Tanaka):

- 60 歳以上のアクティヴなロジシャンのクラブへようこそ!!
  (Welcome to the club of active logicians over 60!!)

The Axiom of Choice over ZF

- The Axiom of Choice (AC) is equivalent to many mathematical assertions over ZF e.g.:

  - For any family $\mathcal{F}$ of (not necessarily Hausdorff) compact spaces, the product space $\prod\mathcal{F}$ is also compact (J.L. Kelley, 1950).

  - Every commutative ring with the unit has a maximal ideal (W. Hodges, 1979).

  - For any field F and any linear algebra A over F, there is a linear basis B of A over F (A. Blass, 1984).

  - ... and many other mathematical assertions (for further asserionts, e.g. ask Assaf Karagila).

- These equivalence results can be interpreted as the facts suggesting the (mathematical) significance of AC (over ZF).

The Continuum Hypothesis over ZFC

- The Continuum Hypothesis (CH) is also known to be equivalent to many mathematical statements over ZFC such as:

  - There is an uncountabe collection $\mathcal{F}$ of analytic (complex) functions s.t. the set $\{f(z) : f \in \mathcal{F}\}$ is countable for every $z \in \mathbb{C}$ (Erdős, 1964)

  - $\mathbb{R}$ can be decomposed into countably many sets X_n, $n \in \omega$ s.t. each X_n is linearly independent over $\mathbb{Q}$ (Erdős and Kakutani, 1943).

- CH also implies many mathematical theorems like:

  - There are functions $f : [0,1]^2 \to [0,1]$ s.t. both $\int_0^1 \int_0^1 f(x,y)dxdy$ and $\int_0^1 \int_0^1 f(x,y)dydx$ exist but they are different.

Reflection of non-metrizability

Theorem 1. (Alan Dow, 1988) If an uncountable compact space X is non-metrizable then there is a non-metrizable subspace of X of cardinality $\aleph_1$.

Very rough sketch of the proof: Take sufficiently closed (more precisely: internally unbounded) elementary submodel $M \prec \mathcal{H}(\chi)$ of cardinality $\aleph_1$ with $X \in M$. Then $X \cap M$ is non metrizable. $\square$

Theorem 2. For any regular cardinal κ there is a topological space X which is not metrizable but all subspaces of X of cardinality $< \kappa$ are metrizable.

Proof: Let $X = \kappa + 1$ where κ is discrete and $\{\kappa + 1 \setminus \alpha : \alpha \in \kappa\}$ forms the nbhd base of κ. $\square$

Reflection of non-metrizability (2/4)

Does the reflection of non-metrizability hold for locally compact spaces?

- The answer is independent:
- $V = L$ produces a counter example (folklore).
- Axiom R (a consequence of Martin's Maximum) implies the reflection of non-metrizability for locally compact spaces (Z. Balogh, 2002).

Reflection of non-metrizability (3/4)

- The reflection (down to size $\aleph_1$) of non-metrizability for locally compact spaces can be characterized by a set-theoretic principle called FRP (Fodor-type reflection principle):

FRP: For any regular uncountable κ, for any stationary $S \subseteq \kappa$ consisting of ordinals of cofinality ω and for any $g : S \to [\kappa]^{\leq\aleph_0}$, there is $I \in [\kappa]^{\aleph_1}$ s.t.

(1) $cf(\sup I) = \omega_1$

(2) $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$

(3) for any regressive $f : S \cap I \to \kappa$ with $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ s.t. $f^{-1}{}''\{\xi^*\}$ is stationary in $\sup(I)$.

Reflection of non-metrizability (4/4)

FRP: For any regular uncountable κ, for any stationary $S \subseteq \kappa$ consisting of ordinals of cofinality ω and for any $g : S \to [\kappa]^{\leq\aleph_0}$, there is $I \in [\kappa]^{\aleph_1}$ s.t.

(1) $cf(\sup I) = \omega_1$

(2) $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$

(3) for any regressive $f : S \cap I \to \kappa$ with $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ s.t. $f^{-1}{}''\{\xi^*\}$ is stationary in $\sup(I)$.

Theorem 3. (S.F., I. Juhász, L. Soukup, Z. Szenttmiklóssy and T. Usuba, 2010)
FRP implies the reflection of non-metrizability of a locally compact space down to a subspace of cardinality $\leq \aleph_1$.

Theorem 4. (S.F., H. Sakai, L. Soukup and T.Usuba) The leflection in Theorem 3 implies FRP.

FRP is equivalent to many "mathematical" statements

- FRP is equivalent to the following assertions over ZFC:
- ▷ For every uncountable locally compact space X, if X is non-metrizable then there is a non-metrizable subspace of X of carinality $\aleph_1$.
- ▷ If an uncountable T_1-space X is not left separated then there is a subspace of X of cardinality $\aleph_1$ which is not left separated.
- ▷ For any graph G if the coloring number of G is uncountable then there is a subgraph of G of cardinality $\aleph_1$ with uncountable coloring number.
- ▷ If an uncountable Boolean algebra B is not openly generated then there are stationarily many subalgebras of B of cardinality $\aleph_1$ which are not openley generated (SF+A.Rinot, 2011).

Some more facts about FRP

- FRP implies the total failure of square principle.
- FRP implies the Singular Cardinal Hypothesis (actually it even implies Shelah's Strong Hypothesis, (S.F.+A.Rinot, T.Usuba)).
- Rado's Conjecture (If a tree is not special then there is an uncountable subtree which is not special) implies FRP.
- Martin's Maximum also implies FRP.
- FRP is preserved by c.c.c. extension. Hence FRP is conistent with large continuum.

▷ All of these statements are of course true for the mathematical statements equivalent to FRP.

A base theory

- Let ZFC_ω be the theory obtained by replacing the Axiom of Replacing by the statement:
  - For a class function $\mathcal{F}$, $\mathcal{F}''x$ is a set for any countable x.
- Many of the known equivalence over ZFC are still valid over ZFC_ω.
- ZFC_ω may be regarded as the theory of the superuniverse of the set generic multiverses. I shall discuss more about this in RIMS set theory meeting in the next week.

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