Models of Weak König's Lemma

Tin Lok Wong

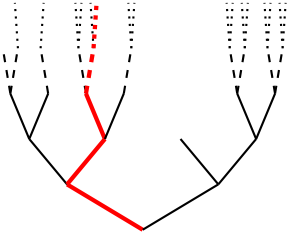
Kurt Gödel Research Center for Mathematical Logic
Vienna, Austria

Joint work with Ali Enayat (Gothenburg)

10 September, 2015

Financial support from FWF Project P24654-N25 is acknowledged.

This talk



Weak König's Lemma (WKL)

Every infinite 0–1 tree has an infinite branch.

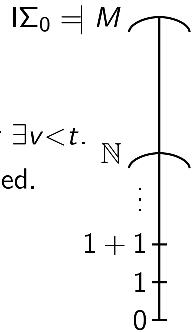
Plan

1. Motivation
2. Self-embeddings
3. Set-extensions
4. Conclusion

models of WKL	$\approx$	coded subsets in end extensions

First-order arithmetic

$n \in \mathbb{N}$



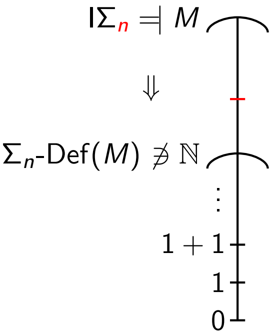
- $\mathscr{L}_{\mathrm{I}} = \{0, 1, +, \times, <\}$.
- A quantifier is *bounded* if it is of the form $\forall v{<}t$ or $\exists v{<}t$.
- An $\mathscr{L}_{\mathrm{I}}$-formula is Δ_0 if all its quantifiers are bounded.
- $\Sigma_n = \{\exists \bar{v}_1\ \forall \bar{v}_2\ \cdots\ \mathrm{Q}\bar{v}_n\ \theta(\bar{v}, \bar{x}) : \theta \in \Delta_0\}$.
- The dual is called Π_n.
- A formula is Δ_n if it is both Σ_n and Π_n.
- $\mathrm{I}\Sigma_n$ consists of some basic axioms (PA^-) and for every $\theta \in \Sigma_n$,
$$\theta(0) \wedge \forall x\ \big(\theta(x) \to \theta(x+1)\big) \to \forall x\ \theta(x).$$
- $\mathrm{B}\Sigma_{n+1}$ consists of the axioms of $\mathrm{I}\Sigma_0$ and for every $\theta \in \Sigma_{n+1}$,
$$\forall a\ \big(\forall x{<}a\ \ \exists y\ \theta(x, y) \to \exists b\ \forall x{<}a\ \ \exists y{<}b\ \ \theta(x, y)\big).$$
- exp asserts the totality of $x \mapsto 2^x$.

Theorem (Paris–Kirby 1978; Parsons 1970; Parikh 1971)

$\mathrm{I}\Sigma_{n+1} \vdash \mathrm{B}\Sigma_{n+1} \vdash \mathrm{I}\Sigma_n$ for all $n \in \mathbb{N}$; and $\mathrm{I}\Sigma_1 \vdash \mathrm{exp}$ but $\mathrm{B}\Sigma_1 \nvdash \mathrm{exp}$.

Cuts and end extensions

$n \in \mathbb{N}$



Definition

Let $I, M \models I\Sigma_0$. Say I is a *cut* of M, or M is an *end extension* of I, if $I \subseteq M$ and

$$\forall i \in I \;\; \forall m \in M \setminus I \;\; i \leqslant m.$$

In this case, write $I \subseteq_e M$.

- $I\Sigma_n$ consists of some basic axioms (PA^-) and for every $\theta \in \Sigma_n$,

$$\theta(0) \wedge \forall x \; \big(\theta(x) \to \theta(x+1)\big) \to \forall x \; \theta(x).$$

Proposition (folklore)

(1) $\mathbb{N}$ is a cut of every model of $I\Sigma_0$, called the *standard cut*.

(2) If $M \not\cong \mathbb{N}$ and $M \models I\Sigma_n$, then $\underbrace{\mathbb{N} \text{ is } \textit{not } \Sigma_n\text{-definable in } M}_{\text{saturation condition}}$.

M is *nonstandard*

Second-order arithmetic

- $\mathscr{L}_{\mathrm{II}} = \{0, 1, +, \times, <, \in\}$ has a *number sort* and a *set sort*.
- A quantifier is *bounded* if it is of the form $\forall v{<}t$ or $\exists v{<}t$.
- $\Delta^0_0, \Sigma^0_n, \Pi^0_n, \Delta^0_n$ are defined as in $\mathscr{L}_{\mathrm{I}}$.
- Formulas in $\bigcup_{n\in\mathbb{N}} \Sigma^0_n$ are called *arithmetical*.
- Δ^0_1-CA stands for the Δ^0_1-comprehension axiom.
- $\mathsf{RCA}_0 = \mathsf{I}\Sigma^0_1 + \Delta^0_1\text{-CA}$. $\mathsf{RCA}^*_0 = \mathsf{B}\Sigma^0_1 + \exp + \Delta^0_1\text{-CA}$.
- $\mathsf{WKL}_0 = \mathsf{RCA}_0 + \mathsf{WKL}$. $\mathsf{WKL}^*_0 = \mathsf{RCA}^*_0 + \mathsf{WKL}$.
- If $M \models \mathsf{I}\Sigma_1$, then $(M, \Delta_1\text{-Def}(M)) \models \mathsf{RCA}_0 + \neg\mathsf{WKL}$.
- If $M \models \mathsf{B}\Sigma_1 + \exp$, then $(M, \Delta_1\text{-Def}(M)) \models \mathsf{RCA}^*_0 + \neg\mathsf{WKL}$.
- If $M \models \mathsf{PA} = \bigcup_{n\in\mathbb{N}} \mathsf{I}\Sigma_n$, then $(M, \mathrm{Def}(M)) \models \mathsf{WKL}_0$.

Theorem (Harrington 1977)

If $\sigma = \forall X\ \varphi(X)$ where φ is arithmetical, then

$$\mathsf{WKL}_0 \vdash \sigma \quad \Rightarrow \quad \mathsf{RCA}_0 \vdash \sigma.$$

Coded sets

Let $M \subseteq_{\mathrm{e}} K \models \mathrm{I}\Sigma_0$.

- Say $c \in K$ *codes* $S \subseteq M$ if
$$S = \{x \in M : \text{the } x\text{th prime divides } c\}.$$
- Denote by $\mathrm{Cod}(K/M)$ the set of all $S \subseteq M$ coded in K.

Theorem (Scott 1962)

If $M \subsetneq_{\mathrm{e}} K \models \mathrm{I}\Sigma_0$ and $M \models \exp$, then $(M, \mathrm{Cod}(K/M)) \models \mathrm{WKL}_0^*$.

Theorem (Enayat–W)

The following are equivalent for a countable $(M, \mathscr{X}) \models \mathrm{I}\Sigma_0^0 + \exp$.

(a) $(M, \mathscr{X}) \models \mathrm{WKL}_0^*$.

(b) $\mathscr{X} = \mathrm{Cod}(K/M)$ for some $K \supsetneq_{\mathrm{e}} M$ satisfying $\mathrm{I}\Sigma_0$.

Self-embeddings (pointwise fixing an initial segment)

Theorem (H. Friedman 1973; Dimitracopoulos–Paris 1988)

For every countable nonstandard $M \models \mathrm{I}\Sigma_1$, there exist $I \subsetneq_{\mathrm{e}} M$ and an isomorphism $M \to I$.

Theorem (Ressayre 1987)

The following are equivalent for all countable $M \models \mathrm{I}\Sigma_0$.

(a) $M \ncong \mathbb{N}$ and $M \models \mathrm{I}\Sigma_1$.

(b) For every $a \in M$, there exist $I \subsetneq_{\mathrm{e}} M$ and an isomorphism $M \to I$ which fixes all $x < a$.

Theorem (Tanaka 1997)

The following are equivalent for all countable $(M, \mathscr{X}) \models \mathrm{I}\Sigma^0_0$.

(a) $M \ncong \mathbb{N}$ and $(M, \mathscr{X}) \models \mathrm{WKL}_0$.

(b) For every $a \in M$, there exist $I \subsetneq_{\mathrm{e}} M$ and an isomorphism $(M, \mathscr{X}) \to (I, \mathrm{Cod}(M/I))$ which fixes all $x < a$.

Self-embeddings

closure of Σ_1 under Boolean operations and bounded quantification

Proposition (folklore)

If $M \ncong \mathbb{N}$ and $M \models \mathsf{I}\Sigma_1$, then $\mathbb{N}$ is *not* $\Delta_0(\Sigma_1)$-definable in M.

Theorem (Dimitracopoulos–Paris 1988)

The following are equivalent for a countable $M \models \mathsf{I}\Sigma_0 + \exp$.

(a) $M \cong I$ for some $I \subsetneq_{\mathrm{e}} M$.

(b) $M \models \mathsf{B}\Sigma_1$ and $\mathbb{N}$ is *not* parameter-free $\Delta_0(\Sigma_1)$-definable in M.

Theorem (Enayat–W)

The following are equivalent for a countable $(M, \mathscr{X}) \models \mathsf{I}\Sigma^0_0 + \exp$.

(a) $(M, \mathscr{X}) \cong (I, \mathrm{Cod}(M/I))$ for some $I \subsetneq_{\mathrm{e}} M$.

(b) $(M, \mathscr{X}) \models \mathsf{WKL}^*_0$ and $\mathbb{N}$ is *not* parameter-free $\Delta_0(\Sigma_1)$-definable in M.

not related to $\mathscr{X}$

Tanaka's Conjecture

not true for
$\sigma = \exists X \ \varphi(X)$
in general

Theorem (Harrington 1977)

If $\sigma = \forall X \ \varphi(X)$ where φ is arithmetical, then

$$\mathsf{WKL}_0 \vdash \sigma \quad \Rightarrow \quad \mathsf{RCA}_0 \vdash \sigma.$$

<u>予想 T.</u> 算術式 φ を使って, $\sigma = \forall X \, \exists ! Y \, \varphi(X, Y)$ と表される命題 σ に対して,

$$\mathsf{WKL}_0 \vdash \sigma \quad \Rightarrow \quad \mathsf{RCA}_0 \vdash \sigma.$$

Tanaka's Conjecture (1995)

If $\sigma = \forall X \ \exists ! Y \ \varphi(X, Y)$ where φ is arithmetical, then

$$\mathsf{WKL}_0 \vdash \sigma \quad \Rightarrow \quad \mathsf{RCA}_0 \vdash \sigma.$$

The model theory behind Tanaka's Conjecture

Theorem (Simpson–Tanaka–Yamazaki 2002)

If $\sigma = \forall X\ \exists! Y\ \varphi(X, Y)$ where φ is arithmetical, then

$$\mathsf{WKL}_0 \vdash \sigma \quad \Rightarrow \quad \mathsf{RCA}_0 \vdash \sigma.$$

Harrington:
$\sigma = \forall X\ \varphi(X)$

Lemma (Harrington 1977)

Every countable $(M, \mathscr{X}) \models \mathsf{RCA}_0$ can be extended to $(M, \mathscr{Y}) \models \mathsf{WKL}_0$.

Lemma (Simpson–Tanaka–Yamazaki 2002)

Every countable $(M, \mathscr{X}) \models \mathsf{RCA}_0$ can be extended to $(M, \mathscr{Y}_1), (M, \mathscr{Y}_2) \models \mathsf{WKL}_0$ such that

(a) $\mathscr{Y}_1 \cap \mathscr{Y}_2 = \mathscr{X}$; and

(b) $(M, \mathscr{Y}_1)$ and $(M, \mathscr{Y}_2)$ satisfy the same formulas with parameters from $(M, \mathscr{X})$.

Models of WKL $\approx$ coded subsets in end extensions

- Ressayre, Tanaka: Having an isomorphism onto a proper cut fixing any given initial segment characterizes $\mathsf{I}\Sigma_1$ and WKL_0.
- Dimitracopoulos–Paris, Enayat–W: Having an isomorphism onto a proper cut is a sign of saturation.
- Simpson–Tanaka–Yamazaki: Any countable $(M, \mathscr{X}) \models \mathsf{RCA}_0$ can be extended to $(M, \mathscr{Y}_1), (M, \mathscr{Y}_2) \models \mathsf{WKL}_0$ with minimal intersection such that the same formulas with parameters from $(M, \mathscr{X})$ are satisfied in them.

Questions

(1) Can every $(M, \mathscr{X}) \models \mathsf{RCA}_0^*$ be extended to $(M, \mathscr{Y}) \models \mathsf{WKL}_0^*$?

(2) Scott 1962: Given $(M, \mathscr{X}) \models \mathsf{WKL}_0$, can one always find $K \supsetneq_{\mathrm{e}} M$ satisfying $\mathsf{I}\Sigma_0$ such that $\mathrm{Cod}(K/M) = \mathscr{X}$?

(3) Can every countable $(M, \mathscr{X}) \models \mathsf{RCA}_0^*$ be extended to $(M, \Delta^0_1\text{-}\mathrm{Def}(M, A)) \models \mathsf{RCA}_0^*$ for some $A \subseteq M$?