Set-theoretic geology with large cardinals

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Definability of the ground model

Theorem 1 (Laver, Woodin)

In a forcing extension $V[G]$ of the universe V, the ground model V is a 1st order definable class in $V[G]$; there is a 1st order formula $\varphi(x, y)$ and a set $r \in V$ such that

$$x \in V \iff V[G] \models \varphi(x, r)$$

Uniform definability of the ground models

Theorem 2 (Fuchs-Hamkins-Reitz)

There is a 1st order formula $\Phi(x, y)$ such that

1. For every set r, the class $W_r = \{x : \Phi(x, r)\}$ is a transitive model of ZFC containing all ordinals, and W_r is a ground of the universe V, that is, there is a poset $\mathbb{P} \in W_r$ and a $(W_r, \mathbb{P})$-generic G with $W_r[G] = V$.
2. For every transitive model $M \subseteq V$ of ZFC, if M is a ground of V, then there is $r \in M$ such that $W_r = M$.

In ZFC, we can consider the structure of all grounds $\{W_r : r \in V\}$.

Now the study of the structure of the grounds is called

Set-Theoretic Geology.

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How many grounds?

What are the problems?

- The order structure of the grounds.
  - How many grounds are there?

Definition 3

1. We say that V has set-many grounds if there is a set X such that $\{W_r : r \in X\}$ is the collection of all grounds: $\forall r \exists s \in X\,(W_r = W_s)$
2. If the cardinality of X is κ, then V has at most κ many grounds.
3. If there is no such a set X, V has proper class many grounds.
4. If $W_r = V$ for every r, then V has no proper grounds.

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Observation 4

1. The constructible universe L does not have a proper ground.
2. In a forcing extension of L, there is a proper ground but there are set many grounds.

Theorem 5 (Reitz, Fuchs-Hamkins-Reitz)

1. There is a class forcing $\mathbb{P}$ which forces that "There is no proper ground".
2. There is a class forcing $\mathbb{Q}$ which forces that "There are proper class many grounds", moreover it forces that "every ground has a proper ground".

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Observation 6

A class forcing $\mathbb{P}$ which forces "there is no proper grounds" preserves almost all large cardinal.

Corollary 7

"No proper grounds" and "there are set many grounds" are consistent with almost all large cardinals.

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Observation 8

A class forcing $\mathbb{Q}$ which forces "there are proper class many grounds" can preserve supercompact cardinals, but it does not preserve large large cardinals, cardinals stronger than the supercompact cardinals in some senses.

Examples of large large cardinals:

- An infinite cardinal δ is extendible if for every $\alpha > \delta$ there is $\beta > \alpha$ and an elementary embedding $j : V_\alpha \to V_\beta$ such that critical point of j is δ (that is, $j(\gamma) = \gamma$ for $\gamma < \delta$ but $j(\delta) > \delta$) and $\alpha < j(\delta)$.
- An infinite cardinal δ is superhuge if for every cardinal $\lambda > \delta$, there are a (definable) transitive model M of ZFC and a (definable) elementary embedding $j : V \to M$ such that the critical point of j is δ, $\lambda < j(\delta)$, and M is closed under $j(\delta)$-sequences.

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Question 9

Is the statement "there are proper class many grounds" consistent with large large cardinals?

An answer is NO! It is inconsistent with some large large cardinal.

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Definition 10

An infinite cardinal δ is super-supercompact (WANT: better name!) if for every cardinal $\lambda > \delta$, there are a (definable) transitive model M of ZFC and a (definable) elementary embedding $j : V \to M$ such that

1. The critical point of δ, and $\lambda < j(\delta)$.
2. M is closed under $j(\lambda)$-sequences.

Observation 11

1. If δ is 2-huge, then there is $\gamma < \delta$ with $V_\delta \models$"γ is super-supercompact".
2. If δ is super-supercompact, then δ is extendible and superhuge, so super-supercompact cardinal is a large large cardinal.

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Main result

Theorem 12

Suppose δ is a super-supercompact cardinal. Then for every ground W_r, there are a poset $\mathbb{P} \in W_r$ and an $(W_r, \mathbb{P})$-generic G such that $|\mathbb{P}| < \delta$ and $V = W_r[G]$.
In other words, if δ is a super-supercompact cardinal, then V must be a small forcing extension of each grounds.

Corollary 13

Suppose δ is super-supercompact. Then there are at most δ many grounds.

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Theorem 14 (Hamkins)

Let W_r and W_s are grounds of V, and suppose there are posets $\mathbb{P} \in W_r$, $\mathbb{Q} \in W_s$, $(W_r, \mathbb{P})$-generic G, and $(W_s, \mathbb{Q})$-generic H such that $V = W_r[G] = W_s[H]$.
Let κ be a regular uncountable cardinal. If $|\mathbb{P}| < \kappa$, $|\mathbb{Q}| < \kappa$, $\mathcal{P}(\kappa) \cap W_r = \mathcal{P}(\kappa) \cap W_s$, then $W_r = W_s$.

Corollary 15

For each ground W_r, fix a poset $\mathbb{P} \in W_r$ and a $(W_r, \mathbb{P})$-generic G with $V = W_r[G]$. Let

1. $\kappa_r :=$ the minimum regular cardinal κ with $|\mathbb{P}| < \kappa$.
2. $P_r := \mathcal{P}(\kappa) \cap W_r$.

Then the correspondence $W_r \mapsto \langle \kappa_r, P_r \rangle$ is injective.

Suppose δ is super-supercompact. Then there are at most δ many grounds.

For each ground W_r, there is a poset $\mathbb{P}_r$ with size $< \delta$ and a $(W_r, \mathbb{P}_r)$-generic G with $V = W_r[G]$. The correspondence $W_r \mapsto \langle \kappa_r, P_r \rangle$ is an injection from the grounds to V_δ, so there are at most δ many grounds.

Question 16

Can super-supercompact be replaced by extendible or superhuge?

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Indestructibility of large cardinals

Theorem 17 (Laver)

It is consistent that δ is supercompact and the supercompactness of δ is preserved by every δ-directed closed forcing.

A poset $\mathbb{P}$ is δ-directed closed if for every $X \subseteq \mathbb{P}$ with size $< \delta$, if X is lower directed then X has a lower bound in $\mathbb{P}$.

Destructibility of large large cardinals

Theorem 18 (Bagaria-Hamkins-Tsarprounis-Usuba)

If δ is a large large cardinal (e.g., superhuge, extendible) then every non-trivial δ-closed forcing must destroy the large large cardinal property of δ.

Theorem 19

1. If there is some poset $\mathbb{Q}$ which forces that "δ is super-supercompact", then $\mathbb{Q}$ is forcing equivalent to a poset of size $< \delta$ and δ is super-supercompact in V.
2. super-supercompact cardinal is extremely destructible: if δ is super-supercompact and $\mathbb{P}$ is a poset which is not forcing equivalent to a poset of size $< \delta$, then $\mathbb{P}$ must destroy the super-supercompactness of δ.

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Core of the grounds

Definition 20

1. The mantle $\mathbf{M}$ is the intersection of all grounds, $\mathbf{M} = \bigcap_r W_r$.
2. The generic mantle $g\mathbf{M}$ is the intersection of all grounds of all generic extensions.

Theorem 21 (Fuchs-Hamkins-Reitz)

1. $\mathbf{M}$ and $g\mathbf{M}$ are definable classes and $g\mathbf{M} \subseteq \mathbf{M}$.
2. $g\mathbf{M}$ is a transitive model of ZF containing all ordinals.
3. $g\mathbf{M}$ is a forcing invariant class.

On the other hand, it is unknown if the following always hold:

1. $g\mathbf{M}$ satisfies AC.
2. $\mathbf{M}$ is a model of ZF(C).
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The mantle may be a ground

If V is a concrete model such as L, $L[A], K$, then we know that $\mathbf{M} = g\mathbf{M}$ and $\mathbf{M}$ is a model of ZFC.

However if large cardinals exist, then "V is $L, L[A], K$" is impossible

Definition 22

A set x is ordinal definable if there is a 1st order formula $\varphi(y, a_0, \ldots, a_n)$ and ordinals $\alpha_0, \ldots, \alpha_n$ such that

$$x = \{y : \varphi(y, \alpha_0, \ldots, \alpha_n)\}.$$

A set x is hereditarily ordinal definable if every element of the transitive closure of x is ordinal definable.

HOD is the class of all hereditarily ordinal definable sets.

It is known that **HOD** is a definable transitive model of ZFC containing all ordinals.

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Fact 23

$V = \textbf{HOD}$ is consistent with almost all large cardinals.
In particular, $V = \textbf{HOD}$ is consistent with the existence of a super-supercompact cardinal.

Proposotion 24

Suppose $V = \mathbf{HOD}$ (or $V = \mathbf{HOD}_{\{A\}}$ for some set A of ordinals).

1. Then $\mathbf{M} = g\mathbf{M}$ is a model of ZFC.
2. Every two grounds have a common ground.
3. If there is a super-supercompact cardinal, then $\mathbf{M}$ is a ground of V. Hence $\mathbf{M}$ is the minimum ground of V. Consequently,
   - 3.1 $\mathbf{M}[G] = V$ for some $G \subseteq \mathbb{P} \in V_\delta$.
   - 3.2 V has strictly less than δ many grounds.

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Is the assumption $V = \mathbf{HOD}$ necessary to establish these results? (For 3., AC in $g\mathbf{M}$ is sufficient.)

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