Reverse Mathematics and Equilibria of Continuous Games

a joint work with NingNing Peng and Weiguang Peng

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Definition of continuous games

Definition 1 (RCA_0)**.** *A continuous game (with n players) is a $2n$-tuple $\langle A_1 \ldots, A_n; f_1 \ldots, f_n \rangle$ where*

1. *each A_i is a nonempty compact metric space,*
2. *each f_i is a continuous function form $\prod A_i \to \mathbb{R}$ with a modulus of uniform continuity.*

From now, we'd like to consider an extension of Nash's theorem by Glicksberg on continuous games.

Fix a continuous game $G = \langle A_1 \ldots, A_n; f_1 \ldots, f_n \rangle$.

We call a probability measure μ over A_i, a mixed strategy of player i. Recall that in Reverse Mathematics, a probability measure μ over a compact metric space X is defined as a positive linear functional on $C(X)$ with $\mu(1)$ where 1 is a constant function whose output is 1.

Of course, we define a mixed profile by a finite sequence of mixed strategies for all players. But to define an expected value of a pay-off function, we have to think of it as the product of measures.

Recall again that $C(X)$ is defined as a complete separable metric space given from a countable set of "polynomials" on X with sup-norm. In fact, to have the same space, we can use diverse notions of "polynomial". So, to define a product measure, we use the following version of S-W theorem.

Lemma 1 (RCA_0)**.** *Let X be a compact metric space with an evidence $\langle\langle x_{ij} : i \leq n_j\rangle : j \in \mathbb{N}\rangle$ for compactness. Assume that a sequence $\langle p_n : n \in \mathbb{N}\rangle$ of $C(X)$ satisfies the following:* (1) $p_0 = 1$,

(2) $\{p_n : n \in \mathbb{N}\}$ *is cloesd under* $+, \cdot$ *and* $r\cdot$ *for any* $r \in \mathbb{Q}$.

(3) *there exists a function* $h : \mathbb{N}^2 \to \mathbb{N}$ *such that for each* i *and* j, $p_{h(i,j)}$ *satisfies* $0 \leq p_{h(i,j)} \leq 1$, $p_{h(i,j)} = 1$ *on* $\overline{B(x_{ij}, 2^{-j})}$ *and* $p_{h(i,j)} = 0$ *on* $B(x_{ij}, 2^{-j+1})^c$.

Then, there exists an effective correspondence between $\hat{A} = \langle\mathbb{N}, d_A\rangle$ *and* $C(X)$ *where* $d_A(n,m) = ||p_n - p_m||_\infty$

Let P_i be the "polynomials" over $\mathbb{Q}$ from basic functions on A_i. Then $P = \{\sum_{j<m} p_{1j} \cdots p_{nj}$: each $p_{ij} \in P_i\}$ can be regarded as satisfying the conditions of Lemma 1 for $C(A)$. $(A = \prod A_i)$

So we may assume that $C(A) = \langle P, || \cdot ||_\infty \rangle$. Then, given a sequence $\langle \mu_1, \ldots \mu_n \rangle$ such that each μ_i is a probability measure on A_i, we can define the product measure μ on A by

$$\mu(p_1 \cdots p_n) = \prod_i \mu_i(p_i),$$

for any $p_i \in P_i$ with the canonical extension.

We identify a mixed profile as a product measure given from it.

Let μ be a mixed profile of G. Then we define the expected value of the payoff functions f_i by

$$f_i(\mu) = \int f_i d\mu \text{ (i.e., } = \mu(f_i))$$

.

Note that $\mu_{-i} = \langle \mu_j : j \neq i \rangle$ is a vector of mixed strategies for all players except i. We sometimes write (μ_i, μ_{-i}) instead of μ.

Definition 2 (RCA_0)**.** *A mixed Nash equilibrium of a continuous game* $\langle A_1, \ldots, A_n; f_1 \ldots, f_n \rangle$ *is a mixed profile* μ^* *such that for all* i, $1 \leq i \leq n$, $f_i(\mu^*) \geq f_i(\mu_i, \mu^*_{-i})$ *for any mixed strategy* μ_i *of player* i.

Glicksberg's theorem

Glicksberg's theorem is proved in ACA_0.

Theorem 2 (ACA_0)**.** *Every continuous game has a mixed Nash equilibrium.*

Our proof is essentially based on Ozdaglar's proof on Glicksberg's theorem in his lecture notes. In the proof, the following fact is the most important.

Theorem 3. *The following assertions are pairwise equivalent over* RCA_0*.*

(1) ACA_0*.*

(2) Any sequence $\langle \mu_n : n \in \mathbb{N} \rangle$ *of probability measures on a compact metric space* X *has a weak convergent subsequence* $\langle \mu_{n_k} : k \in \mathbb{N} \rangle$*, that is, there exists a probability measure* μ *such that* $\mu(f) = \lim_{k \to \infty} \mu_{n_k}(f)$ *for all* $f \in C(X)$*.*

proof. (1) $\to$ (2). Let X be a compact metric space and $C(X)$ coded by $(P, ||\cdot||_\infty)$. Let $\langle p_i : i \in \mathbb{N}\rangle$ be an enumeration of the elements of P. For each $n \in \mathbb{N}$ and $\sigma \in 2^{<\mathbb{N}}$, we define a closed interval I^i_σ by

$$I^i_\sigma = [\,(\sum_{j<lh(\sigma)} \frac{\sigma(j)}{2^j} - 1)||p_i||_\infty, (\sum_{j<lh(\sigma)} \frac{\sigma(j)}{2^j} - 1 + \frac{1}{2^{Ih(\sigma)-1}})||p_i||_\infty\,].$$

Take any sequence $\langle \mu_n : n \in \mathbb{N}\rangle$ of probability measures on X. Then we have sequences $\langle n_k : k \in \mathbb{N}\rangle$ and $\langle \sigma^i_k : i, k \in \mathbb{N}, i \leq k\rangle$ such that

$$\mathrm{lh}(\sigma^i_k) = k,\ \sigma^i_{k+1} \succeq \sigma^i_k$$

and $\mu_{n_k}(p_i) \in I^i_{\sigma^i_k}$ for all $i \leq k$.

Then, $|I^i_{\sigma^i_k}| = 2^{-k+1}||p_i||_\infty$ and $I^i_{\sigma^i_{k+1}} \subset I^i_{\sigma^i_k}$ for all $i \leq k$.

So, we can get a probability measure μ by

$$\mu(p) = \lim_{k\to\infty} \mu_{n_k}(p),\ for\ all\ p \in P.$$

Thus, we have a sequence $\langle \mu_{n_k} : k \in \mathbb{N} \rangle$ which converges to μ.

The converse direction (2)→(1) is easy.

Let $\langle c_n : n \in \mathbb{N} \rangle$ be a sequence on $[0, 1]$. We may only show that it has a convergent subsequence.

Define a probability measure μ_n on $[0, 1]$ by $\mu_n(f) = f(c_n)$.

From our assumption (2), it has a subsequence

$$\mu_{n_k} \to \mu$$

for some probability measure μ. Take $f \in C([0, 1])$ with $f(x) = x$.

Then, $\langle \mu_{n_k}(f) : k \in \mathbb{N} \rangle$ is a subsequence of $\langle c_n : n \in \mathbb{N} \rangle$ which converges to $\mu(f)$. $\Box$.

The rest of the proof of Theorem 2 is like this: Take any continuous game $G = \langle A_1 \ldots, A_n; f_1 \ \ldots, f_n \rangle$.

1. By compactness of $A = \prod_i A_i$, we can find a sequence of "essentially finite games" $\langle G_m : m \in \mathbb{N} \rangle$ such that $|f_i^m(\sigma) - f_i(\sigma)| \leq 2^{-m}$, where $G_m = \langle A_1 \ldots, A_n; f_1^m \ \ldots, f_n^m \rangle$
2. By the sequence version of Nash's theorem, there exists a sequence $\langle \mu_m : m \in \mathbb{N} \rangle$ of mixed strategies such that each μ_m is Nash equilibrium of G_m.
3. Using Theorem 3, pick up a subsequence $\langle \mu_{m_k} : k \in \mathbb{N} \rangle$ converging to μ.
4. By 1, $f_i(a_i, (\mu_{m_k})_{-i}) \leq f_i(\mu_{m_k}) + 2^{-m_k+1}$ for all $a_i \in A_i$. As $k \to \infty$, $f_i(a_i, (\mu)_{-i}) \leq f_i(\mu)$, that is, μ is a Nash equilibrium.

Remark.

1. An "essentially finite game" may not be continuous. Indeed, it is coded by using simple functions with arithmetical sets in ACA_0.
2. The sequence version of Nash's theorem can be proved in WKL_0. But, when we apply it to "essentially finite" games, so far, we require ACA_0 because we have to construct a probability measure on A from a mixed strategies of finite games on the way of the proof.
3. I don't know if Glicksberg's theorem implies ACA_0 or not.

Recall our version of Browder Fixed point theorem:

Theorem 4. *The following assertion is equivalent to* WKL_0 *over* RCA_0. *Let* $\hat{A}$ *be a nonempty compact convex "subspace" of* $\mathbb{R}^n$. *Let an open subset* T *of* $\hat{A} \times \hat{A}$ *satisfy the following conditions:*

(i) for any $x \in \hat{A}$*, there exists* $y \in \hat{A}$ *such that* $(x, y) \in T$*,*

(ii) for any $x, y_1, y_2 \in \hat{A}$ *and any positve reals* r_1 *and* r_2*, if* $(x, y_1), (x, y_2) \in T$ *and* $r_1 + r_2 = 1$*, then* $(x, r_1 y_1 + r_2 y_2) \in T$.

Then there exists $x \in \hat{A}$ *such that* $(x, x) \in T$.

Definition 3 (RCA_0). *A pure Nash equilibrium of a continuous game* $\langle A_1, \ldots, A_n; f_1 \ldots, f_n \rangle$ *is a profile* $a^* \in A$ *such that for all* i, $1 \le i \le n$, $f_i(a^*) \ge f_i(a_i, a^*_{-i})$ *for all* $a_i \in A_i$.

A continuous function $f : [0,1] \to \mathbb{R}$ is concave if for any $x, y \in [0,1]$ $tf(x) + (1-t)f(y) \le f(tx + (1-t)y)$.

Then, we have the following result.

Theorem 5. *The following is provable in* WKL_0. *Let* $G = \langle [0,1] \ldots, [0,1]; f_1 \ldots, f_n \rangle$ *be a continuous game. Assume that any* $f_i(\cdot, x_{-i})$ *is concave. Then there exists a pure Nash equilibrium.*

References

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Thank you so much!!