Effective Reducibility for Smooth and Analytic Equivalence Relations on a Cone

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Joint Work with
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1 Invariant descriptive set theory:

2 Computable structure theory:
Invariant descriptive set theory: classification of classification problems of mathematical structures such as:

- Isomorphism relation on countable Boolean algebras.
- Isomorphism relation on countable $p$-groups.
- Isometry relation on Polish metric spaces.
- Linear isometry relation on separable Banach spaces.
- Isomorphism relation on separable $C^*$-algebras.

Key notion: Borel reducibility among equivalence relations on Borel spaces.

Computable structure theory:
1 Invariant descriptive set theory: classification of classification problems of mathematical structures such as:
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   - Linear isometry relation on separable Banach spaces.
   - Isomorphism relation on separable $C^*$-algebras.

Key notion: Borel reducibility among equivalence relations on Borel spaces.

2 Computable structure theory: classification of classification problems of *computable structures* such as:
   - Isomorphism relation of computable trees.
   - Isomorphism relation of computable torsion-free abelian grps
   - Bi-embeddability relation of computable linear orders.

Key notion: *computable reducibility* among equivalence relations on *represented spaces*.
- \((X, \delta)\) is a **represented space** if \(\delta : \subseteq \mathbb{N}^\mathbb{N} \rightarrow X\) is a partial surjection.
- A point \(x \in X\) is **computable** if it has a computable name, that is, there is a computable \(p \in \delta^{-1}\{x\}\).
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**Example**

1. The **space of countable \(L\)-structures** is represented:
   For a countable relational language \(L = (R_i)_{i \in \mathbb{N}}\), each countable \(L\)-structure \(K\) with domain \(\subseteq \omega\) is identified with its atomic diagram \(D(K) = \bigoplus_{i \in \mathbb{N}} R^K_i \in 2^\omega\).
   For a class \(\mathbb{K}\) of countable \(L\)-structures with \(\delta : D(K) \hookrightarrow K\), \((\mathbb{K}, \delta)\) forms a represented space.

2. Polish spaces, second-countable \(T_0\) space are represented.

3. Much more generally, every \(T_0\) space with a countable cs-network has a “universal” representation \(\delta\), i.e., for any representation \(\delta'\), there is a continuous map \(g\) such that \(\delta' = \delta \circ g\).
(X, δ) is a **represented space** if δ :⊆ N^N → X is a partial surjection.

A point x ∈ X is **computable** if it has a computable name, that is, there is a computable p ∈ δ^{-1}{x}.

The e-th computable point of X = (X, δ) is denoted by Φ^X_e.

Let E and F be equivalence relations on represented spaces X and Y, respectively. We say that E ≤_{eff} F if there is a partial computable function f :⊆ N → N such that for all i, j ∈ N with Φ^X_i, Φ^X_j ∈ dom(δ_X),

$$\Phi^X_i E \Phi^X_j \iff \Phi^Y_{f(i)} F \Phi^Y_{f(j)}.$$  

Let E and F be equivalence relations on Borel spaces X and Y, respectively. We say that E ≤_{B} F if there is a Borel function f : X → Y such that for all x, y ∈ X,

$$x E y \iff f(x) F f(y).$$
Today’s Theme

“Effective reducibility on a cone”
i.e., the oracle-relativized version of effective reducibility.
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i.e., the **oracle-relativized version** of effective reducibility.

- The **oracle relativization** of a computability-theoretic concept sometimes has applications in other areas of mathematics which does **NOT** involve any notion concerning computability:
  - (Gregoriades-K., K.-Ng) the Shore-Slaman join theorem / The Louveau separation theorem $\xrightarrow{\sim}$ a decomposition theorem for Borel measurable functions in descriptive set theory.
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  - (Gregoriades-K., K.-Ng) the Shore-Slaman join theorem / The Louveau separation theorem \(\iff\) a decomposition theorem for Borel measurable functions in descriptive set theory.
  - (K.-Pauly) Turing degree spectrum / Scott ideals (\(\omega\)-models of \(WKL\)) \(\iff\) a refinement of R. Pol’s solution to Alexandrov’s problem in infinite dimensional topology.
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- The oracle relativization of a computability-theoretic concept sometimes has applications in other areas of mathematics which does NOT involve any notion concerning computability:
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  - (K.-Pauly) Turing degree spectrum / Scott ideals $\rightsquigarrow$ a construction of linearly non-isometric (ring non-isomorphic, etc.) examples of Banach algebras of real-valued Baire $n$ functions on Polish spaces.
Let $E$ and $F$ be equivalence relations on represented spaces $X$ and $Y$, respectively. We say that $E \leq^{\text{cone eff}} F$ if there is a partial computable function $f : \subseteq \mathbb{N} \to \mathbb{N}$ such that $(\exists r \in 2^\omega)(\forall z \geq_T r)$ for all $i, j \in \mathbb{N}$ with $\Phi^z_{X,i}, \Phi^z_{X,j} \in \text{dom}(\delta_X)$,

$$\Phi^z_{X,i} E \Phi^z_{X,j} \iff \Phi^{z,Y}_{f(i)} F \Phi^{z,Y}_{f(j)}.$$

$$E \leq_c F \implies E \leq_B F$$

$\downarrow$

$$E \leq^{\text{cone eff}} F \implies E \leq^{\text{cone hyp}} F$$

- $E$ is said to be **analytic $\leq^{\text{cone eff}}$-complete** if $F \leq^{\text{cone eff}} E$ for any analytic equivalence relation $F$.

- $E$ is said to be **$\leq^{\text{cone eff}}$-intermediate** if
  - $E$ is not analytic $\leq^{\text{cone eff}}$-complete,
  - and there is no Borel eq. relation $F$ such that $E \leq^{\text{cone eff}} F$. 

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Takayuki Kihara (UC Berkeley)  
**Effective Reducibility on a Cone**
The Vaught Conjecture (1961)

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- (The $L_{\omega_1 \omega}$-Vaught conjecture) The number of countable models of an $L_{\omega_1 \omega}$-theory is at most countable or $2^{\aleph_0}$.

- (Topological Vaught conjecture) The number of orbits of a continuous action of a Polish group on a standard Borel space is at most countable or $2^{\aleph_0}$.
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Fact (Becker 2013; Knight and Montalbán)

Suppose that there is no $\mathcal{L}_{\omega_1 \omega}$-axiomatizable class of countable structures whose isomorphism relation is $\leq^{\text{cone}}_{\text{eff}}$-intermediate then, the $\mathcal{L}_{\omega_1 \omega}$-Vaught conjecture is true.

Indeed, if there is no $\leq^{\text{cone}}_{\text{eff}}$-intermediate orbit equivalence relation then, the topological Vaught conjecture is true.
The differences of $\leq_B$ and $\leq_{\text{cone}}^{\text{eff}}$ among non-Borel orbit eq. relations:

For Borel reducibility (H. Friedman and Stenley 1989):

- The isomorphism relation on an $L_{\omega_1 \omega}$-axiomatizable class of countable structure \textit{CANNOT} be analytic $\leq_B$-complete.
- Moreover, the isomorphism relation on countable torsion abelian groups is \textit{NOT} $\leq_B$-complete even among isomorphism relations on classes of countable structures.

For computable reducibility (Fokina, S. Friedman, et al. 2012):

- The isomorphism relations on computable graphs, torsion-free abelian groups, fields (of a fixed characteristic), etc. are $\leq_{\text{eff}}$-complete analytic equivalence relations.
- The isomorphism relation on computable torsion abelian groups is also a $\leq_{\text{eff}}$-complete analytic equivalence relation.
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In this talk, we focus on the differences of $\leq_B$ and $\leq_{\text{cone eff}}$ among

- non-Borel non-orbit analytic equivalence relations,

- and smooth equivalence relations.
Non-orbit analytic equivalence relations:

\[ xE_{\text{wo}}y : \iff \text{either } x, y \notin \text{WO or } x \text{ and } y \text{ are isomorphic as w.o.} \]

\[ xE_{\text{ck}}y : \iff \omega_1^x = \omega_1^y \text{ holds.} \]

Fact

- (Gao) \( E_{\text{wo}} \) and \( E_{\text{ck}} \) are \( \leq_B \)-incomparable.
- (Coskey-Hamkins 2011) \( E_{\text{wo}} \) and \( E_{\text{ck}} \) are \( \leq_{\text{ITTM}} \)-bireducible.
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Theorem

- \( E_{ck} \leq_{\text{eff}} \text{cone} E_{wo} \).
- If \( V = L \), then \( E_{ck} <_{\text{eff}} \text{cone} E_{wo} \).
Non-orbit analytic equivalence relations:

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Theorem

- $E_\text{ck} \leq_{\text{eff}}^{\text{cone}} E_\text{wo}$.
- If $V = L$, then $E_\text{ck} \prec_{\text{eff}}^{\text{cone}} E_\text{wo}$.

Conjecture

If $x^\#$ exists for any real $x$, then $E_\text{ck} \equiv_{\text{eff}}^{\text{cone}} E_\text{wo}$.
\( T(A, B) \): the tree of partial isomorphisms between \( A \) and \( B \).

For partial orders \( A = (A, \leq_A) \) and \( B = (B, \leq_B) \) with \( A, B \subseteq \omega \) \( \sigma \oplus \tau \in T(A,B) \) iff

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**Lemma (Upper Bound)**

$\alpha < \beta < \omega_1$: ordinals.

$A \in \text{LO}$ s.t. $\text{otype}(A) = \alpha + \lambda$, where $\lambda$ has no least element.

$B \in \text{LO}$ s.t. $\text{otype}(B) = \beta + \theta$ for a linear order $\theta$.

Then, $\text{rank}(T(A, B)) \leq \omega^{\alpha+2}$.

$\beta$ is $\alpha$-closed if $(\forall \gamma < \alpha)(\forall \delta < \beta) \delta + \gamma < \beta$.

**Lemma (Lower Bound)**

$\alpha, \beta < \omega_1$: ordinals, $\beta$ is $\omega^\alpha$-closed, $c \in \omega$

$A \in \text{WO}$ s.t. $\text{otype}(A) = \omega^\alpha \cdot c$.

$B \in \text{WO}$ s.t. $\text{otype}(B) = \beta$.

Then, $\text{rank}(T(A, B)) \geq \omega \cdot \alpha$. 

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Lemma

\( \mathcal{A} \): a well order s.t. \( \text{otype}(\mathcal{A}) = \alpha \).

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$$\text{rank}(T(\mathcal{A}, \mathcal{B})) \leq \sup_k \text{rank}T(\mathcal{A} \upharpoonright k, \mathcal{B} \upharpoonright n) + 2n + 1.$$
Lemma (Upper Bound)

$A \in \text{LO}$ s.t. $\text{otype}(A) = \alpha + \lambda$, where $\lambda$ has no least element.

$B \in \text{LO}$ s.t. $\text{otype}(B) = \beta + \theta$ for $\beta > \alpha$ and linear $\theta$.

Then, $\text{rank}(T(A, B)) \leq \omega^{\alpha+2}$. 

\[\begin{array}{ccc}
A & B & T(A, B) \\
\uparrow \lambda & \uparrow \theta & \\
\alpha \times & \beta &
\end{array}\]
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- \( \text{rank}(T(\mathcal{A}, \mathcal{B})) \leq \sup_k (\sup_m \text{rank} T(\mathcal{A} \upharpoonright l_k, \mathcal{B} \upharpoonright m) + l_k) + 2n + 1. \)
- \( \text{rank}(\mathcal{A} \upharpoonright l_k, \mathcal{B} \upharpoonright m) \leq \omega^{\alpha_m + 1}, \text{ where } \alpha_m := \text{otype}(\mathcal{B} \upharpoonright m) < \alpha. \)
\( \mathcal{A} \in \text{WO s.t. otype}(\mathcal{A}) = \omega^\alpha. \)
\( \mathcal{B} \in \text{WO s.t. otype}(\mathcal{B}) = \beta \text{ s.t. } (\forall \gamma < \omega^\alpha)(\forall \delta < \beta) \delta + \gamma < \beta. \)
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- If $\alpha$ is limit, choose an increasing seq. $\alpha_0 < \alpha_1 < \cdots \rightarrow \alpha$.
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\[ A \in \mathrm{WO} \text{ s.t. } \text{o-type}(A) = \omega^\alpha. \]
\[ B \in \mathrm{WO} \text{ s.t. } \text{o-type}(B) = \beta \text{ s.t. } (\forall \gamma < \omega^\alpha)(\forall \delta < \beta) \delta + \gamma < \beta. \]
Then, \[ \text{rank}(T(A, B)) \geq \omega \cdot \alpha. \]

- If \( \alpha \) is limit, choose an increasing seq. \( \alpha_0 < \alpha_1 < \cdots \to \alpha \).
- If \( \alpha \) is successor, we use \( \omega^{(\alpha-1)} \cdot j \) instead of \( \omega^{\alpha_j} \).
- \( A_0 \simeq B_0 \simeq \omega^\alpha \cdot (c - 1) \) and \( A_2 \simeq B_2 \simeq \gamma_1 \).
- \( A_1^j \simeq \omega^{\alpha_j}, B_1 \simeq \gamma_0; A_3 \simeq \omega^\alpha, B_3 \) is \( \omega^\alpha \)-closed.
(L, <L): a linear order
Define the linear order \(\omega^L = (\text{CNF}(L), \leq_{\omega^L})\) as follows:

1. \(\text{CNF}(L) = \{(\lambda_i, c_i)_{i<n} \in (L \times \omega)^{<\omega} : (\forall i) \lambda_{i+1} <_L \lambda_i\},\)
2. \((\lambda_i, c_i)_{i<n} \leq_{\omega^L} (\lambda'_j, c'_j)_{j<m} \iff (\exists k < m, n) \text{ s.t.}\)
   - \((\forall i < k) \lambda_i = \lambda'_i\) and
   - \(\lambda_k <_L \lambda'_k\) or \((\lambda_k = \lambda'_k\) and \(c_i \leq c^*_i\)).

Inductively define \(\exp^0(L) = L\) and \(\exp^{n+1}(L) = \omega^\exp^n(L)\).
Define \(\varepsilon(L)\) by \(\sum_{n \in \omega} \exp^n(L)\).
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   \begin{align*}
   & (\forall i < k) \lambda_i = \lambda'_i \text{ and} \\
   & \lambda_k <_L \lambda'_k \text{ or } (\lambda_k = \lambda'_k \text{ and } c_i \leq c^*_i).\end{align*}\)

- If \(L\) is not well-ordered, then so is \(\omega^L\).
- \(L \in \text{WO}, (\lambda_i, c_i)_{i < n} \approx \sum_{i < n} \omega^{\lambda_i} \cdot c_i.\)
- \(L \in \text{WO}, \text{oype}(L) = \alpha \Rightarrow \text{oype}(\omega^L) = \omega^\alpha.\)

Inductively define \(\exp^0(L) = L\) and \(\exp^{n+1}(L) = \omega^{\exp^n(L)}.\)

Define \(\varepsilon(L)\) by \(\sum_{n \in \omega} \exp^n(L)\).
Proof of $E_{ck} \leq_{cone}^{eff} E_{wo}$

1. $H^x$: Harrison’s pseudo well order relative to $x$ whose order type is $\omega^x_1 \cdot (1 + \eta)$.

2. Given $z$ and $x \leq_T z$, define $f(x) := \varepsilon(KB(T(H^x, H^z)))$.
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3. If $\omega^x_1 = \omega^z_1$, then $H^x$ is isomorphic to $H^z$.
   - $\Rightarrow$ the KB ordering on $T(H^x, H^z)$ is not well-ordered; therefore, $f(x) \notin WO$. 

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3. If $\omega_1^x = \omega_1^z$, then $H^x$ is isomorphic to $H^z$.
   - $\Rightarrow$ the KB ordering on $T(H^x, H^z)$ is not well-ordered; therefore, $f(x) \notin WO$.

4. If $\omega_1^x < \omega_1^z$, $\omega \cdot \omega_1^x \leq \text{rank}(T(H^x, H^z)) \leq \omega_1^{\omega_1^x+2}$.
   - $\varepsilon(\omega_1 \cdot \omega_1^x)$ is isomorphic to $\varepsilon(\omega_1^{\omega_1^x+2})$.
   - Hence, $\text{otype}(\varepsilon(KB(T(H^x, H^z)))) = \varepsilon(\omega_1^x)$.
   - Thus, $\omega_1^x = \omega_1^y < \omega_1^z$ implies $f(x) \approx f(y) \approx \varepsilon(\omega_1^x) = \varepsilon(\omega_1^y)$. 

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4. If $\omega_1^x < \omega_1^z$, $\omega \cdot \omega_1^x \leq \text{rank}(T(\mathcal{H}^x, \mathcal{H}^z)) \leq \omega_1^{\omega_1^x+2}$.
   - $\varepsilon(\omega \cdot \omega_1^x)$ is isomorphic to $\varepsilon(\omega_1^{\omega_1^x+2})$.
   - Hence, $\text{otype}(\varepsilon(\text{KB}(T(\mathcal{H}^x, \mathcal{H}^z)))) = \varepsilon(\omega_1^x)$.
   - Thus, $\omega_1^x = \omega_1^y < \omega_1^z$ implies $f(x) \approx f(y) \approx \varepsilon(\omega_1^x) = \varepsilon(\omega_1^y)$.

5. Thus, $\omega_1^x = \omega_1^y \iff f(x), f(y) \notin \text{WO}$ or $f(x) \approx f(y)$. 

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Proof of “$V = L$ implies $E_{ck} \lesscong^{\text{cone eff}} E_{wo}$”

- Weitkamp (1982): If $V$ is a generic extension of $L$, then the following set contains no Turing cone:

  $$\{ x \in 2^\omega : \omega_1^x \text{ is a recursively inaccessible ordinal} \}.$$

- Given $r$, choose $z \geq_T r$ s.t. $\omega_1^z$ is NOT rec. inaccessible.
Proof of “$V = L$ implies $E_{ck} \leq_{\text{cone eff}} E_{wo}$”

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- Given $r$, choose $z \geq_T r$ s.t. $\omega_1^z$ is NOT rec. inaccessible.

- Then, for any admissible ordinal $\alpha \leq \omega_1^z$, there is a $\Pi^1_1(z)$ set $P_\alpha \subseteq 2^\omega$ such that
  
  $$\{ x \leq_T z : \omega_1^x = \alpha \} = P_\alpha \cap \{ x \in 2^\omega : x \leq_T z \}.$$
Proof of “$V = L$ implies $E_{\text{ck}} \lesssim^\text{cone eff} E_{\text{wo}}$”

• Weitkamp (1982): If $V$ is a generic extension of $L$, then the following set contains no Turing cone:

$$\{ x \in 2^\omega : \omega^x_1 \text{ is a recursively inaccessible ordinal} \}.$$ 

• Given $r$, choose $z \geq_T r$ s.t. $\omega^z_1$ is NOT rec. inaccessible.

• Then, for any admissible ordinal $\alpha \leq \omega^z_1$, there is a $\Pi^1_1(z)$ set $P_\alpha \subseteq 2^\omega$ such that

$$\{ x \leq_T z : \omega^x_1 = \alpha \} = P_\alpha \cap \{ x \in 2^\omega : x \leq_T z \}.$$ 

• Thus, there is no $z$-effective reduction from $E_{\text{wo}}$ to $E_{\text{ck}}$ since $\{ x \leq z : x \notin \text{WO} \}$ is $\Sigma^1_1(z)$-complete.
Non-orbit analytic equivalence relations:

\[ x E_{wo} y : \Longleftrightarrow \text{either } x, y \not\in \text{WO or } x \text{ and } y \text{ are isomorphic as w.o.} \]
\[ x E_{ck} y : \Longleftrightarrow \omega^x_1 = \omega^y_1 \text{ holds.} \]

Fact

- (Gao) \( E_{wo} \) and \( E_{ck} \) are \( \leq_B \)-incomparable.
- (Coskey-Hamkins 2011) \( E_{wo} \) and \( E_{ck} \) are \( \leq_{ITTM} \)-bireducible.

Theorem

- \( E_{ck} \leq_{\text{cone eff}} E_{wo} \).
- If \( V = L \), then \( E_{ck} \prec_{\text{cone eff}} E_{wo} \).

Conjecture

If \( x^\# \) exists for any real \( x \), then \( E_{ck} \equiv_{\text{cone eff}} E_{wo} \).
Smooth Equivalence Relations

$\Delta_X$: the equality $(X, =)$ on a topological space $X$. 
$\leq_B$ ($\leq_c$, resp.): Borel (continuous, resp.) reducibility.

1. $\Delta_X \equiv_B \Delta_Y$ whenever $X$ and $Y$ are uncountable standard Borel spaces. In particular, $\Delta_{2^\omega} \equiv_B \Delta_{I^n} \equiv_B \Delta_{I^\omega}$

2. $\Delta_{2^\omega} <_c \Delta_{I} <_c \Delta_{I^2} <_c \cdots <_c \cdots <_c \Delta_{I^n} <_c \Delta_{I^{n+1}} < \Delta_{I^\omega}$.

Theorem

1. $\Delta_{2^\omega} <_{\text{cone eff}} \Delta_{I} <_{\text{cone eff}} \Delta_{I^2}$.

2. $\Delta_{I^3} \equiv_{\text{cone eff}} \Delta_{I^4} \equiv_{\text{cone eff}} \cdots \equiv_{\text{cone eff}} \Delta_{I^n} \equiv_{\text{cone eff}} \Delta_{I^{n+1}} \equiv_{\text{cone eff}} \Delta_{I^\omega}$. 
Remark

- $\Delta_X \leq_{\text{eff}} \Delta_Y$ iff $\exists$ a Markov computable injection $f : X_{\text{cpt}} \rightarrow Y_{\text{cpt}}$.
- (Kreisel-Lacombe-Shoenfield) $f : (\omega^\omega)_{\text{cpt}} \rightarrow (\omega^\omega)_{\text{cpt}}$ is Markov computable iff it is computable in the sense of TTE.
- (de Brecht) $X$ has a total admissible representation iff $X$ is quasi-Polish.
- Hence, whenever $X$ and $Y$ are quasi-Polish, $\Delta_X \leq_{\text{eff}} \Delta_Y$ iff there is a TTE-computable injection $f : X_{\text{cpt}} \rightarrow Y_{\text{cpt}}$. 
Proof Idea of $\Delta_I^n \leq^{\text{cone eff}} \Delta_{I^3}$

1. The $n$-dimensional sphere $S^n$ is not an absolute extensor for $I^{n+1}$.

2. $S^n$ is an absolute extensor for a normal space $X$ if and only if the covering dimension of $X$ is at most $n$.

3. If the covering dimension of a separable metric space $X$ is $\leq n$, then it is embedded into the $n$-dimensional Nöbeling space $N^n \subseteq I^{2n+1}$.
Proof Idea of $\Delta_\mu^n \leq_{\text{cone eff}} \Delta_\beta$

1. The $n$-dimensional sphere $S^n$ is \textit{not} an absolute extensor for $I^{n+1}$.

   \begin{itemize}
   \item[(★)] It is computably FALSE!:
   The 1-sphere $S^1$ is a computable absolute extensor for $I^{n+1}_{\text{cpt}}$.
   \end{itemize}

2. $S^n$ is an absolute extensor for a normal space $X$
   \iff the covering dimension of $X$ is at most $n$.

3. If the covering dimension of a separable metric space $X$ is $\leq n$, then it is embedded into the $n$-dimensional Nöbeling space $N^n \subseteq I^{2n+1}$. 
Proof Idea of $\Delta^n \leq^\text{cone}_{\text{eff}} \Delta^\beta$

1. The $n$-dimensional sphere $S^n$ is not an absolute extensor for $I^{n+1}$.

   $(\star)$ It is computably FALSE!:
The 1-sphere $S^1$ is a computable absolute extensor for $I_{\text{cpt}}^{n+1}$.
   (constructive counterexample to Brouwer’s fixed point thm.)

2. $S^n$ is an absolute extensor for a normal space $X$
   \iff the covering dimension of $X$ is at most $n$.

3. If the covering dimension of a separable metric space $X$ is $\leq n$, then
   it is embedded into the $n$-dimensional Nöbeling space $N^n \subseteq I^{2n+1}$.
Proof Idea of $\Delta I^n \leq_{\text{cone eff}} \Delta I^3$

1. The $n$-dimensional sphere $S^n$ is not an absolute extensor for $I^{n+1}$.

   (⋆) It is computably FALSE!
   The 1-sphere $S^1$ is a computable absolute extensor for $I_c^{n+1}$.
   (constructive counterexample to Brouwer's fixed point thm.)

2. $S^n$ is an absolute extensor for a normal space $X$
   $\iff$ the covering dimension of $X$ is at most $n$.

   (⋆) It is computably TRUE:
   $S^n$ is a cpt. absolute extensor for a cpt. normal space $X_{cpt}$
   $\iff$ the cpt. covering dimension of $X_{cpt}$ is at most $n$.

3. If the covering dimension of a separable metric space $X$ is $\leq n$, then it is embedded into the $n$-dimensional Nöbeling space $N^n \subseteq I^{2n+1}$. 

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Proof Idea of $\Delta^m \leq^\text{cone eff} \Delta_\beta$

1. The $n$-dimensional sphere $S^n$ is not an absolute extensor for $I^{n+1}$.
   
   (★) It is computably FALSE!:
   The 1-sphere $S^1$ is a computable absolute extensor for $I^{n+1}_{\text{cpt}}$.
   (constructive counterexample to Brouwer’s fixed point thm.)

2. $S^n$ is an absolute extensor for a normal space $X$
   $\iff$ the covering dimension of $X$ is at most $n$.

   (★) It is computably TRUE:
   $S^n$ is a cpt. absolute extensor for a cpt. normal space $X_{\text{cpt}}$
   $\iff$ the cpt. covering dimension of $X_{\text{cpt}}$ is at most $n$.
   Hence, the computable covering dimension of $I^n_{\text{cpt}}$ is at most 1!

3. If the covering dimension of a separable metric space $X$ is $\leq n$, then it is embedded into the $n$-dimensional Nöbeling space $N^n \subseteq I^{2n+1}$.
Proof Idea of $\Delta_1^m \leq_{\text{cone eff}} \Delta_1$

1. The $n$-dimensional sphere $S^n$ is not an absolute extensor for $I^{n+1}$.

   ($\star$) It is computably FALSE!
   The 1-sphere $S^1$ is a computable absolute extensor for $I_\text{cpt}^{n+1}$.
   (constructive counterexample to Brouwer’s fixed point thm.)

2. $S^n$ is an absolute extensor for a normal space $X$
   $\iff$ the covering dimension of $X$ is at most $n$.

   ($\star$) It is computably TRUE:
   $S^n$ is a cpt. absolute extensor for a cpt. normal space $X_{\text{cpt}}$
   $\iff$ the cpt. covering dimension of $X_{\text{cpt}}$ is at most $n$.
   Hence, the computable covering dimension of $I_\text{cpt}^n$ is at most 1!

3. If the covering dimension of a separable metric space $X$ is $\leq n$, then
   it is embedded into the $n$-dimensional Nöbeling space $N^n \subseteq I_{\text{cpt}}^{2n+1}$.

   ($\star$) It is computably TRUE:
Proof Idea of $\Delta^n \leq^\text{cone eff} \Delta^3$ 

1. The $n$-dimensional sphere $S^n$ is not an absolute extensor for $I^{n+1}$.
   
   (★) It is computably FALSE!:
   
   The 1-sphere $S^1$ is a computable absolute extensor for $I^{n+1}$.
   (constructive counterexample to Brouwer’s fixed point thm.)

2. $S^n$ is an absolute extensor for a normal space $X$
   
   $\iff$ the covering dimension of $X$ is at most $n$.

   (★) It is computably TRUE:
   
   $S^n$ is a cpt. absolute extensor for a cpt. normal space $X_{\text{cpt}}$
   
   $\iff$ the cpt. covering dimension of $X_{\text{cpt}}$ is at most $n$.
   
   Hence, the computable covering dimension of $I^n_{\text{cpt}}$ is at most 1! 

3. If the covering dimension of a separable metric space $X$ is $\leq n$, then it is embedded into the $n$-dimensional Nöbeling space $N^n \subseteq I^{2n+1}$.

   (★) It is computably TRUE:
   
   Hence, $I^n_{\text{cpt}}$ is computably embedded into $N^1 \subseteq I^3$. 

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