The completeness theorem, WKL_0 and the origins of Reverse Mathematics

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Simpson (1999) on Reverse Mathematics

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Table: Foundational programs and the five basic systems.

RCA_0	constructivism	Bishop
WKL_0	finitistic reductionism	Hilbert
ACA_0	predicativism	Weyl, Feferman
ATR_0	predicative reductionism	Friedman, Simpson
Π^1_1-CA_0	impredicativity	Feferman et al.

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Thus we can expect this book and other Reverse Mathematics studies to have a substantial impact on the philosophy of mathematics.

1999, p. 42

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We identify as ***ordinary*** or ***non-set-theoretic*** that body of mathematics which is prior to or independent of the introduction of abstract set-theoretic concepts. We have in mind such branches as geometry, number theory, calculus, differential equations, real and complex analysis, countable algebra, the topology of complete separable metric spaces, mathematical logic, and computability theory. 2009, p. 1-2

Friedman (1974) on Reverse Mathematics

The questions underlying the work presented here on subsystems of second order arithmetic are the following. What are the proper axioms to use in carrying out proofs of particular theorems, or bodies of theorems, in mathematics? What are those formal systems which isolate the essential principles needed to prove them? . . . ¶ . . .

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In our work, two principal themes emerge.

I) When the theorem is proved from the right axioms, the axioms can be proved from the theorem . . .

II) **Much more is needed to define explicitly hard-to-define [sets] of integers than merely to prove their existence.** An example of this theme which we consider is that the natural axioms needed to define explicitly nonrecursive sets of natural numbers prove the consistency of the natural axioms needed to prove the existence of nonrecursive sets of natural numbers. 1974, p. 235

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5) This aspect of WKL came to light during the metamathematical investigation of the Gödel (1929/1930) Completeness Theorem.
6) As such, WKL_0 bears both on the philosophical significance of the Completeness Theorem and more generally on the status of Hilbert's dictum "consistency implies existence".

Outline

I) Review

II) What is a "set existence axiom"?

III) History of WKL_0 and the completeness theorem (1899-1974):

Frege, Hilbert, Löwenheim, Skolem, J. & D. König, Gödel, Hilbert & Bernays, Maltsev, Lindenbaum, Tarski, Hasenjaeger, Henkin, Kleene, Beth, Kreisel, Wang, Montague, Scott, Shoenfield, Jockusch & Soare, Friedman, Kriesel & Simpson & Mints

IV) Some philosophical observations and guarded conclusions:

- existence simpliciter vs conditional existence
- consistency $\Rightarrow$ existence ?
- ontological commitment *de dicto* and *de re*

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The five basic subsystems subsystems

- Subsystems:
  - $\mathrm{RCA}_0 = \mathrm{PA}^- + \mathrm{Ind}(\Sigma^0_1) + \Delta^0_1\text{-}\mathrm{CA}_0$
  - $\mathrm{WKL}_0 = \mathrm{RCA}_0 + \mathrm{WKL}$
  - $\mathrm{ACA}_0 = \mathrm{RCA}_0 + \mathrm{Ind}(\mathcal{L}_2) + \mathcal{L}_1\text{-}\mathrm{CA}$
  - $\mathrm{ATR}_0 = \mathrm{ACA}_0 + \mathrm{ATR}$
  - $\Pi^1_1\text{-}\mathrm{CA}_0 = \mathrm{RCA}_0 + \mathrm{Ind}(\mathcal{L}_2) + \Pi^1_1\text{-}\mathrm{CA}$
- $\mathrm{RCA}_0 \subsetneq \mathrm{WKL}_0 \subsetneq \mathrm{ACA}_0 \subsetneq \mathrm{ATR}_0 \subsetneq \Pi^1_1\text{-}\mathrm{CA}_0$
- Each of the five systems is **finitely axiomatizable**.

On the formulation of WKL in $\mathcal{L}_2$

The following definitions are made in RCA_0:

- A ***tree*** is a set $T \subseteq \mathbb{N}^{<\mathbb{N}}$ which is closed under initial segs.
- T is ***finitely branching*** if each $\sigma \in T$ has only *finitely many* immediate successors $\tau = \sigma^\frown\langle n\rangle$, ***binary branching*** if each $\sigma \in T$ has at most *two* successors, and ***0-1*** if $T \subseteq \{0,1\}^{<\mathbb{N}}$.
- A *path* through T is $g : \mathbb{N} \to \mathbb{N}$ such that $g[n] \in T$, $\forall n \in \mathbb{N}$.
- Three arithmetical forms of König's Infinity Lemma:

(KL) $\forall T(\text{Finitely-Branching-Tree}(T)\ \&\ \text{Infinite}(T) \Rightarrow \exists g(g \text{ a path through } T))$

(BKL) $\forall T(\text{Binary-Branching-Tree}(T)\ \&\ \text{Infinite}(T) \Rightarrow \exists g(g \text{ a path through } T))$

(WKL) $\forall T(\text{0-1-Tree}(T)\ \&\ \text{Infinite}(T) \Rightarrow \exists g(g \text{ a path through } T))$

Statements reversing to WKL over RCA_0

The Infinity Lemma [can be applied in] the most diverse mathematical disciplines, since it often furnishes a useful method of carrying over certain results from the finite to the infinite ... Some applications of the Infinity Lemma are analogous to applications of the Heine-Borel covering theorem. Because of this it seems interesting to remark that, from a certain standpoint, the Infinity Lemma can be thought of as the proper foundation of this covering theorem. König 1927/1936

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- *Heine-Borel Covering Lemma, Peano existence lemma, Brouwer fixed point theorem.*
- *Every countable consistent set of first-order sentences has a countable model.* (Gödel)
- *If* $\varphi(x)$ *and* $\psi(x)$ *are* Σ^0_1 *s.t.* $\neg\exists x(\varphi(x) \wedge \psi(x))$, *then there is* X *s.t.* $\forall x(\varphi(x) \rightarrow x \in X \wedge \psi(x) \rightarrow x \notin X)$. ($\Sigma^0_1$-Separation)

Existence simpliciter and conditional existence

Orthodox view of “ontological commitment” (Quine 1948):
A theory is committed to those and only those entities to which the bound variables of the theory must be capable of referring in order that the affirmations made in the theory be true.

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  - If S is consistent, then there exists $\mathcal{M} \models \mathrm{S}$.

Comprehension and separation

- Two means of asserting the existence of sets:
  1) By comprehension for a class of formulas Γ:

$$(\Gamma\text{-AC}) \qquad \text{For all } \varphi(x) \in \Gamma \text{ not containing } X \text{ free,}$$
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- Recall the logical form of WKL:

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- WKL does not have the "surface grammar" of either 1) or 2).

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- But if $\mathrm{RCA}_0 \vdash \mathrm{WKL} \leftrightarrow \Gamma\text{-AC}$, then there is a single arithmetical formula $\varphi(x, X)$ s.t.

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- But then $\mathrm{RCA}_0 \vdash \mathrm{WKL}$ by (1). Contradiction.

WKL and separation

Over RCA_0, WKL is equivalent to Σ^0_1-$\textsc{Sep}$.

- Canonical example: Let S be a recursively axiomatized theory.

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$$\forall x, y < lh(t)(\mathrm{Proof}_{\mathrm{S}}(y, x) \to t(x) = 1 \wedge \mathrm{Proof}_{\mathrm{S}}(y, \dot{\neg} x) \to t(x) = 0)$$

- If S is consistent, then T_S is infinite.
- Kleene (1952a): If S is essentially undecidable, then T_{S} has no recursive path.
- But Modulo RCA_0, "T_{S} exists" is a *constructive* claim.
- So modulo, WKL and Σ^0_1-$\textsc{Sep}$ both have the form

If something X exists (constructive),
then something Y exists (possibly non-constructive).

Is WKL a "set existence axiom"? (3)

- Observations:
  1) While WKL is not a set existence principle *simpliciter*, it is a *conditional* set existence principle.
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- Question: Is the import "innocent"?
  - Finitism: no, because there are no infinite trees (or paths).
  - Predicativism: yes, because $\mathrm{ACA}_0 \vdash \mathrm{WKL}$.
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  - Constructivism: complicated, because of the *minimal non-constructivity* of WKL.
- Plan: Use the equivalence of WKL and the Completeness Theorem over RCA_0 to illustrate what's at issue with respect to Hilbert's dictum "consistency implies existence".

Frege vs Hilbert (1899) on model existence

Frege's dictum: "Existence entails consistency."

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Hilbert's dictum: "Consistency entails existence."

I found it very interesting to read this very sentence in your letter, for as long as I have been thinking, writing and lecturing on these things, I have been saying the exact reverse: **if the arbitrarily given axioms do not contradict each other with all their consequences, then they are true and the things defined by the axioms exist**. This is for me the criterion of truth and existence.

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1929, p. 63

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- Jockush & Soare (1972) showed that every recursive theory has a *low* model – i.e. $\texttt{deg}(P_i^M)' = \mathbf{0}'$.

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- Yamazaki (2001) showed that the strong completeness of HPC wrt Kripke models is equivalent over RCA_0 to ACA_0.

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THEOREM 1.7 Suppose $A(X)$ is a Σ^1_1-formula with X as the only free set variable and

$$\mathrm{WKL}_0 \vdash (\exists X)(A(X) \wedge X \text{ is not recursive})$$

then

$$\mathrm{WKL}_0 \vdash \forall Y \exists X (A(X) \wedge X \text{ is not recursive} \wedge \forall n (Y_n \neq X)).$$

Etchemendy (1990) contra Tarksi (1935) on logical truth

- Consider the following sentence:

$$\varphi = (\forall x \forall y \forall z (R(x,y) \land R(y,z) \to R(x,z)) \land \forall x \neg R(x,x)) \to \neg \forall x \exists y R(x,y)$$

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- Question: How far do these commitments extend?

From consistency to non-constructive existence

- Fnitely axiomatizable theories with no recursive models:
  - $\mathrm{EFA} + \neg\mathrm{Con}(\mathrm{EFA})$ (Tennenbaum 1959, MacAloon 1982)
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- Revised Hilbert's dictum:

  "Consistency implies existence non-constructively."

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Completeness formalized in $\mathcal{L}_2$:

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  - If $\langle A_i : i \in \mathbb{N}\rangle$ are non-recursive, then exists ω-model $\mathcal{M} \models \text{WKL}_0$ s.t. $A_i \notin \mathcal{S}_M, \forall i \in \mathbb{N}$.

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- So while Completeness entails non-constructive set existence, it does not require existence of *specific* non-recursive sets.

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- 1.i) is a belief about a specific number (i.e. 33550336).
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- *Perhaps* finitistic reductionists should be understood as being committed *de dicto* to the existence of non-recursive sets but not committed to them *de re*?

Bernays (1950) "Mathematical consistency and existence"

The difficulties to which we have been led here ultimately arise from the fact that the concept of consistency itself is not at all unproblematic. The common acceptance of the explanation of mathematical existence in terms of consistency is no doubt due in considerable part to the circumstance that on the basis of the simple cases one has in mind, one forms an unduly simplistic idea of what consistency (compatibility) of conditions is. One thinks of the compatibility of conditions as something the complex of conditions wears on its sleeve ... In fact, however, the role of the conditions is that they affect each other in functional use and by combination. **The result obtained in this way is not contained as a constituent part of what is given through the conditions.** It is probably the erroneous idea of such inherence that gave rise to the view of the tautological character of mathematical propositions.