

How Many Numbers Can a Lambda-Term Contain?

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Representing numbers in λ -terms

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In this talk we consider simply-typed λ -calculus (types are of the form $\tau \rightarrow \sigma$ constructed out of a base type o). The type of “numbers” is $\mathbb{N} = (o \rightarrow o) \rightarrow o \rightarrow o$. In fact each closed β -normalized term of this type represents some number.

Representing pairs

We can also represent pairs (in terms of type $(\mathbb{N} \rightarrow \mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N}$):

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constructor of pairs:

$$\text{pair} = \lambda n_1. \lambda n_2. \lambda f. f n_1 n_2$$

extractors:

$$\text{ext}_1 = \lambda p. p (\lambda x. \lambda y. x)$$

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In a similar way we can represent triples, quadruples, ...

But (with such natural representation) for tuples of bigger arities we need to use terms of a more complicated type.

Natural question:

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What would it mean?

Of course we can represent k numbers in this way:

$$[(n_1, n_2, \dots, n_k)] = \lambda f. f n_1 (f n_2 (\dots (f n_{k-1} n_k) \dots))$$

but the numbers cannot be extracted...

Representing tuples

Natural question:

Maybe in terms of some type τ we can represent arbitrarily long tuples (arrays) of integers?

It would mean that:

For each k there exist closed terms

$$ktuple : \mathbb{N} \rightarrow \mathbb{N} \rightarrow \dots \rightarrow \mathbb{N} \rightarrow \tau$$

$$kext_1, \dots, kext_k : \tau \rightarrow \mathbb{N}$$

such that

$$\forall i \quad kext_i (ktuple \ n_1 \ n_2 \ \dots \ n_k) \rightarrow_{\beta} n_i$$

Representing tuples

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It would mean that (a weaker statement):

For each k there exist closed terms

$$kext_1, \dots, kext_k : \tau \rightarrow \mathbb{N}$$

and for all $n_1, n_2, \dots, n_k \in \mathbb{N}$ there exists a closed term T of type τ (a representation of this tuple) such that

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Theorem 1

The answer is NO – such type τ does not exist.

Another point of view

Consider the equivalence relation \sim on terms of the same type $\tau \rightarrow \mathbb{N}$:

$K \sim L$ if for each sequence N_1, N_2, \dots of terms of type τ ,

seq. KN_1, KN_2, \dots is bounded \Leftrightarrow seq. LN_1, LN_2, \dots is bounded

e.g. $(\lambda n. n)$ and $(\lambda n. \text{add } n \ n)$ are equivalent.

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For each type τ the relation \sim has finitely many equivalence classes.

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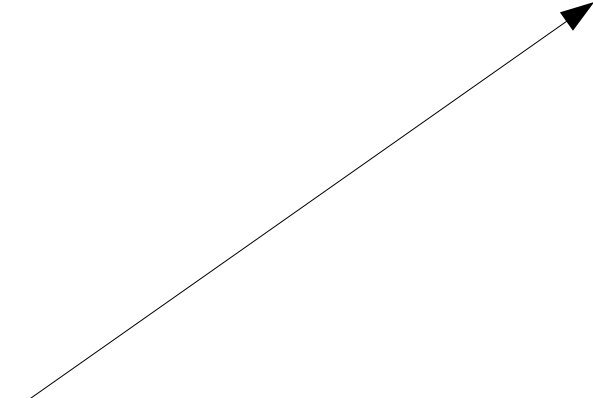
Theorem 1 follows immediately from Theorem 2: the extractors cannot be equivalent, so length of representable tuples is not greater than the number of equivalence classes of \sim .

(Longer tuples cannot be represented even when we allow approximate extraction, up to some error).

Motivation (related work)

A similar theorem turns out to be useful while proving that all higher-order recursion schemes (that is λY -terms) generate more trees than those of them which are “safe”.

“Safety” is a widely considered syntactic restriction, which simplifies some reasonings.



they generate Böhm trees, which are infinite trees



Techniques used

To simplify the analysis we add constants: $\mathbf{0} : o$ and $\mathbf{1+} : o \rightarrow o$.

For each n of type \mathbb{N} , the term $(n \mathbf{1+} \mathbf{0})$ after normalization is of the form

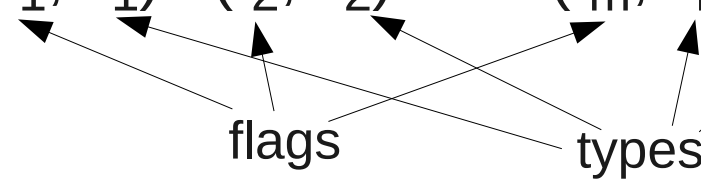
$\mathbf{1+} (\mathbf{1+} (\dots (\mathbf{1+} \mathbf{0}) \dots))$

n

Techniques used

Intersection type system:

- Intersection types refine simple types.
- To a term we assign a pair (flag, type), where $\text{flag} \in \{\text{pr}, \text{np}\}$ (“productive”, “nonproductive”).
- One base type: o .
- The types are of the form $(f_1, \tau_1) \wedge (f_2, \tau_2) \wedge \dots \wedge (f_m, \tau_m) \rightarrow \tau$.



- It will turn out that the equivalence class of \sim depends only on the set of such pairs (flag, type) which can be assigned to a term.

Intersection types

The types are of the form $(f_1, \tau_1) \wedge (f_2, \tau_2) \wedge \dots \wedge (f_m, \tau_m) \rightarrow \tau$.

Diagram illustrating the structure of the type expression: $(f_1, \tau_1) \wedge (f_2, \tau_2) \wedge \dots \wedge (f_m, \tau_m) \rightarrow \tau$. The word "flags" points to the f_i components, and the word "types" points to the τ_i components.

When a term M has such type, it means that if to the argument of the function M we can assign all pairs $(f_1, \tau_1), (f_2, \tau_2), \dots, (f_m, \tau_m)$, then the result has type τ .

Moreover M is required to use its argument in each of these types (we have type $T \rightarrow \tau$ (with $m=0$) when the argument is not used at all).

Thus we know precisely which arguments are used and with which types.

Intersection types

Beside of a type, to a term M we also assign a flag.

Flag “productive” means that M adds something to the resulting value (in addition to the value supported by the arguments):

- the use of $\mathbf{1+}$ is productive (a $\mathbf{1+}$ has to appear in the derivation of a type, which means that it is really used),
- M is productive also when it uses some of its productive arguments more than once (again, we look at the derivation tree).

e.g. $F = (\lambda f. f (f \mathbf{0}))$ is productive, because $(f \mathbf{1+}) = (\mathbf{1+} (\mathbf{1+} \mathbf{0}))$

but $F = (\lambda f. f)$ is nonproductive (even when f is productive),
because $(F (F (F f))) = f$.

To one term we may assign multiple pairs (flag, type).

Techniques used

Step 2: count “how much a term is productive”.

To each typed term M (in fact to a derivation tree for $M:(f,\tau)$) we assign a number $\text{val}(M)$, which counts:

- the number of **1+** nodes in the derivation tree, and
- the number of application nodes KL such that a productive variable is used both in K and in L .

Easy observation – compositionality:

For closed terms it holds $\text{val}(KL)=\text{val}(K)+\text{val}(L)$.

Quite difficult lemma:

For closed terms of base type it holds

$\text{val}(M) \leq$ the number represented by $M \leq 2^{2^2 \dots 2^{\text{val}(M)}}$

the maximal order of a subterm of M

Techniques used

Quite difficult lemma:

For closed terms of base type it holds

$\text{val}(M) \leq \text{the number represented by } M \leq 2^{2^{\dots 2^{\text{val}(M)}}}$ ← the maximal order of a subterm of M

To prove this lemma, we need to:

- isolate closed subterms in M ,
- replace the tower of 2^2 by an appropriately defined $\text{high}(M)$,
- perform the head β -reduction first (closed subterms remain closed), and prove that $\text{val}(M)$ increases and $\text{high}(M)$ decreases.

Proof of the theorem

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$\text{seq. } KN_1, KN_2, \dots \text{ is bounded} \Leftrightarrow \text{seq. } LN_1, LN_2, \dots \text{ is bounded}$

The sequences are almost: (lemma)

$\text{val}(KN_1), \text{val}(KN_2), \dots$ and $\text{val}(LN_1), \text{val}(LN_2), \dots$

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so they differ only by a constant $\text{val}(L) - \text{val}(K)$.

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This is true assuming that we can use the same types for K and L , that is the same type for N_i in both sequences...

Thank you.