JAIST LOGIC WORKSHOP SERIES 2015 - Constructivism and Computability

First application of Proof Mining to Partial Differential Equations; Rates of convergence and metastability for abstract Cauchy problems generated by accretive operators (in J. Math. Anal. Appl. vol. 423, 1089-1112 (2015))

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Origin of proof interpretations

- Hilbert's 2nd problem (1900): Is Peano arithmetic consistent ?
- Gödel (1931) : Impossible to prove the consistency of a theory $\mathcal{T}$ within $\mathcal{T}$.
- Gödel's motivation: obtain a relative consistency proof for HA (and hence for PA).
- Let theories $\mathcal{T}_1$, $\mathcal{T}_2$ with languages $\mathcal{L}(\mathcal{T}_1)$, $\mathcal{L}(\mathcal{T}_2)$. $\mathcal{T}_2$ is consistent relative to $\mathcal{T}_1$ if it can be proved that if $\mathcal{T}_1$ is consistent then $\mathcal{T}_2$ is consistent.
- A theorem $\phi \in \mathcal{L}(\mathcal{T}_1)$ transformed into $\phi' \in \mathcal{L}(\mathcal{T}_2)$; the proof p of ϕ transformed in a proof p' of ϕ'. This often gives new quantitative information. Also: p' using restricted version of the assumptions of ϕ, thus proving a more general result ϕ'.
- Gödel's functional "Dialectica" Interpretation (1958): consistency of PA reduced to a quantifier-free calculus of primitive recursive functionals of finite type.

Proof Mining

G. Kreisel (1950's): *Unwinding of proofs*

'What more do we know if we have proved a theorem by restricted means than if we merely know that it is true ?"

Within the past 20 years, Ulrich Kohlenbach and his collaborators have applied Dialectica and related interpretations to obtain a wide spectrum of results in : approximation theory, ergodic theory, fixed point theory and nonlinear analysis.

Applications described as instances of logical phenomena by general logical metatheorems.

Herbrand normal form - Metastability

- In general, for a Π^0_3 sentence, i.e. of the form

$$A \equiv \forall k \; \exists n \; \forall m \; A_0(k, n, m)$$

  where A_0 is quantifier-free, it is not possible to compute a bound on n.
- However: possible to compute a bound on n for A^H, the *Herbrand normal form* of A;

$$A^H :\equiv \forall k \; \exists n \; A_0(k, n, g(n))$$

  where g is the Herbrand index function (in theories allowing function variables and function quantifiers it would be $A^H :\equiv \forall g, k \; \exists n \; A_0(k, n, g(n))$
- A^H : ineffectively equivalent to A and related to the no-counterexample interpretation of A.

Herbrand normal form - Metastability

An instance in analysis-convergence statements

- considering a statement of the form

$$\lim_{t \to \infty} P(t) = 0$$

- written as

$$\forall k \in \mathbb{N}\ \exists n \in \mathbb{N}\ \forall t \geq n\ (|P(t)| < 2^{-k}),$$

- by considering the metastable version

$$\forall k \in \mathbb{N}\ \forall g : \mathbb{N} \to \mathbb{N}\ \exists n \in \mathbb{N}\ \forall t \in [n, n + g(n)]\ (|P(t)| < 2^{-k}),$$

possible to find a computable bound (*rate of metastability*) $\Phi(k, g, \cdot)$ depending on general uniform bounds on the input data, so that $n \leq \Phi(k, g, \cdot)$.

Application to the Cauchy problem generated by accretive operators

In the following X is a real Banach space with dual X^*. A mapping $A : X \to 2^X$ will be called an operator on X.

Definition

(J. García-Falset, 2005) Let $\phi : X \to [0, \infty)$ continuous with $\phi(0) = 0$, $\phi(x) > 0$ for $x \neq 0$ so that for every sequence (x_n) in X such that $(\|x_n\|)$ is decreasing and $\phi(x_n) \to 0$ as $n \to \infty$, then $\|x_n\| \to 0$. A with $0 \in Az$ is said to be *ϕ-accretive at zero* if

$$\forall (x, u) \in A \ (\langle u, x - z\rangle_+ \geq \phi(x - z)).$$

where: $\langle y, x\rangle_+ := \max\{\langle y, j\rangle : \ j \in J(x)\}$,

$$J(x) := \{j \in X^*; \langle x, j\rangle = \|x\|^2, \ \|j\| = \|x\|\}.$$

Application to the Cauchy problem generated by accretive operators

Preliminaries

Definition

(Kohlenbach, K.-A., 2014) A ϕ-accretive at zero operator A is uniformly ϕ- accretive at zero if $\phi : X \to [0, \infty)$ is of the form $\phi(x) = g(\|x\|)$ where $g : [0, \infty) \to [0, \infty)$ is continuous, $g(0) = 0$ and $g(\alpha) > 0$ for $\alpha \neq 0$.

Definition

(Kohlenbach, K.-A., 2014) Given $\Theta_{(\cdot)}(\cdot) : \mathbb{N} \times \mathbb{N}^* \to \mathbb{N}$, a uniformly ϕ- accretive at zero operator A has a modulus of accretivity Θ if

$$\forall K \in \mathbb{N}^* \, \forall n \in \mathbb{N} \, \forall (x, u) \in A \, (\|x - z\| \in [2^{-n}, K] \to \langle u, x - z \rangle_+ \geq 2^{-\Theta_K(n)}$$

Proposition

(Kohlenbach, K.-A., 2014) Every uniformly ϕ-accretive at zero operator A has a modulus of accretivity Θ .

We introduce in higher generality the property of *uniform accretivity at zero* for an operator $A : D(A) \to 2^X$ with $0 \in Az$ as follows:

Definition

(Kohlenbach, K.-A., 2014) An accretive operator $A : D(A) \to 2^X$ with $0 \in Az$ is called uniformly accretive at zero if

$$\forall k \in \mathbb{N}\ \forall K \in \mathbb{N}^*\ \exists m \in \mathbb{N}\ \forall (x,u) \in A$$

$$(\|x - z\| \in [2^{-k}, K] \to \langle u, x - z\rangle_+ \geq 2^{-m})(*).$$

Any function $\Theta_{(\cdot)}(\cdot) : \mathbb{N} \times \mathbb{N}^* \to \mathbb{N}$ is called a modulus of accretivity at zero for A if $m := \Theta_K(k)$ satisfies $(*)$.

Application to the Cauchy problem generated by accretive operators

Preliminaries

Definition

Let $\mathcal{F} = \{S(t) : \ C \to C, \ t \geq 0\}$ be a family of self-mappings of $C \subseteq X$. $\mathcal{F}$ is said to be a nonexpansive semigroup acting on C if

1. $S(0) = I$, where I is the identity mapping on C,
2. $S(s+t)x = S(s)S(t)x$ for all $s, t \in [0, \infty)$ and $x \in C$,
3. $\|S(t)x - S(t)y\| \leq \|x - y\|$ for all $x, y \in C$ and $t \in [0, \infty)$,
4. $t \to S(t)x$ is continuous in $t \in [0, \infty)$ for each $x \in C$.

Definition

A continuous function $u : [0, \infty) \to C \subseteq X$ is said to be an almost-orbit of $\mathcal{F}$ if

$$\lim_{s \to \infty} \Big(\sup_{t \in [0,\infty)} \|u(t+s) - S(t)u(s)\| \Big) = 0.$$

Application to the Cauchy problem generated by accretive operators

Preliminaries

It is known that the following initial value problem

$$u'(t) + A(u(t)) \ni f(t), t \in [0, \infty)$$

$$u(0) = x$$

where $f \in L^1(0, \infty, X)$. for each $x \in \overline{D(A)}$ has a unique integral solution u so that $u(t) \in \overline{D(A)}$ for all t.
Moreover, it is known that for $x_0 \in \overline{D(A)}$

$$u'(t) + A(u(t)) \ni 0, t \in [0, \infty)$$

$$u(0) = x_0$$

has a unique integral solution given by Crandall- Liggett :

$$u(t) := S(t)(x_0) = \lim_{n \to \infty} (I + \frac{t}{n} A)^{-n}(x_0).$$

Application to the Cauchy problem generated by accretive operators

Theorem

(García-Falset, 2005) If A is an operator on X so that $\forall \lambda > 0$ $(\overline{D(A)} \subseteq R(I + \lambda A))$ that is ϕ-accretive at zero and such that the problem

$$v'(t) + A(v(t)) \ni 0, t \in [0, \infty), v(0) = x_0$$

has a strong solution for each $x_0 \in D(A)$ and $\mathcal{F} := \{S(t) : \overline{D(A)} \to \overline{D(A)} : t \geq 0\}$ is the nonexpansive semigroup generated by $-A$, then every almost-orbit $u : [0, \infty) \to \overline{D(A)}$ of $\mathcal{F}$ is strongly convergent to the zero z of A.

Application to the Cauchy problem generated by accretive operators

Theorem

(Kohlenbach, K.-A., 2014) Same as above with A uniformly accretive at zero with modulus of accretivity Θ . Then every almost-orbit $u : [0, \infty) \to \overline{D(A)}$ of $\mathcal{F}$ is strongly convergent to the zero z of A with rate of metastability $\Psi(k, \bar{g}, B, \Phi, \Theta)$ so that

$$\forall k \in \mathbb{N} \ \forall \bar{g} : \mathbb{N} \to \mathbb{N} \ \exists \bar{n} \leq \Psi(k, \bar{g}, B, \Phi, \Theta)$$

$$\forall t \in [\bar{n}, \bar{n} + \bar{g}(\bar{n})] \ (\|u(t) - z\| < 2^{-k}),$$

where

Application to the Cauchy problem generated by accretive operators

$$\Psi(k, \bar{g}, B, \Phi, \Theta) = \Phi(k+1, g) + h(\Phi(k+1, g))$$

with

$$g(n) := \bar{g}(n + h(n)) + h(n),$$

$$h(n) := (B(n) + 2) \cdot 2^{\Theta_{K(n)}(k+2)+1},$$

$$K(n) := \ulcorner \sqrt{2(B(n)+1)} \urcorner.$$

Here $\Phi : \mathbb{N} \times (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ is a rate of metastability corresponding to a given almost-orbit $u : [0, \infty) \to \overline{D(A)}$ of $\mathcal{F}$, i.e.

$$\forall k \in \mathbb{N}\ \forall g : \mathbb{N} \to \mathbb{N}\ \exists n \leq \Phi(k, g)$$

$$\forall t \in [0, g(n)]\ (\|u(t+n) - S(t)u(n)\| \leq 2^{-k})$$

and $B(n) \in \mathbb{N}$ is any nondecreasing upper bound on $\frac{1}{2}\|u(n) - z\|^2$.

Application to the Cauchy problem generated by accretive operators

We now come to a case where the premise (the information coming from the almost-orbit which before was unknown) can be explicitly witnessed.
This is our central result.

Because the integral solution of

$$u'(t) + A(u(t)) \ni f(t), t \in [0, \infty), u(0) = x$$

where $f(\cdot) \in L^1(0, \infty, X)$ **turns out to be** an almost orbit of $\mathcal{F} := \{S(t) : \overline{D(A)} \to \overline{D(A)} : t \geq 0 \}$ generated by $-A$ via the Crandall-Liggett formula, the following result by García-Falset is actually a corollary of his previous theorem, and analogously our following theorem (our central result) can be seen as a corollary of our metastable theorem above.

Application to the Cauchy problem generated by accretive operators

Theorem

(J. García-Falset, 2005) Let A be an ϕ-accretive at zero operator on X so that $\forall \lambda > 0$ $(\overline{D(A)} \subseteq R(I + \lambda A))$. If

$$v'(t) + A(v(t)) \ni 0, t \in [0, \infty), v(0) = x_0$$

has a strong solution for each $x_0 \in D(A)$, Then for each $x \in \overline{D(A)}$ the integral solution $u(\cdot)$ of

$$u'(t) + A(u(t)) \ni f(t), t \in [0, \infty), u(0) = x$$

where $f(\cdot) \in L^1(0, \infty, X)$ converges strongly to the zero z of A as $t \to \infty$.

Application to the Cauchy problem generated by accretive operators

Theorem

(Kohlenbach, K.-A., 2014) Same as above except that A is a uniformly accretive at zero operator on X with a modulus of accretivity Θ. Then, for each $x \in \overline{D(A)}$ the integral solution $u(\cdot)$ of

$$u'(t) + A(u(t)) \ni f(t), t \in [0, \infty), u(0) = x$$

where $f(\cdot) \in L^1(0, \infty, X)$ $u(\cdot)$ satisfies

$$\forall k \in \mathbb{N}\ \forall \bar{g} : \mathbb{N} \to \mathbb{N}\ \exists \bar{n} \leq \Psi(k, \bar{g}, M, \Theta, B)$$

$$\forall t \in [\bar{n}, \bar{n} + \bar{g}(\bar{n})]\ (\|u(t) - z\| < 2^{-k})$$

with rate of metastability

$$\Psi(k, \bar{g}, M, B, \Theta) = \tilde{g}^{(M\cdot 2^{k+1})}(0) + h(\tilde{g}^{(M\cdot 2^{k+1})}(0))$$

where

$$\tilde{g}(n) := g(n) + n,$$
$$(\; g^{(0)}(k) := k$$
$$g^{(i+1)}(k) := g(g^{(i)}(k)) \;),$$
$$g(n) := \bar{g}(n + h(n)) + h(n),$$
$$h(n) := (B(n) + 2) \cdot 2^{\Theta_{K(n)}(k+2)+1},$$
$$K(n) := \ulcorner \sqrt{2(B(n)+1)} \urcorner,$$

$B(n)$ is a monotone upper bound :$B(n) \geq \frac{1}{2}\|u(n) - z\|^2$,

$$\mathbb{N} \ni M \geq I := \int_0^\infty \|f(\xi)\| d\xi.$$

Thank you !

APPENDIX: Mathematical Definitions

Definition

A continuous function $u : [0, \infty) \to X$ is said to be a *strong solution* of the homogeneous problem if it is Lipschitz on every bounded subinterval of $[0, \infty)$, a.e. differentiable on $[0, \infty)$, $u(t) \in D(A)$ a.e., $u(0) = x_0$ and $u'(t) + A(u(t)) \ni 0$ for almost every $t \in [0, \infty)$.

Definition

A continuous function $u : [0, \infty) \to X$ is an integral solution of the (non)homogeneous problem if $u(0) = x$ and for $s \in [0, t]$ and $(w, y) \in A$

$$\|u(t) - w\|^2 - \|u(s) - w\|^2 \leq 2 \int_s^t \langle f(\tau) - y, u(\tau) - w \rangle_+ d\tau.$$