Generalized Goodstein sequences

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OUTLINE

1. Introduction
   - An example for PRA
   - Another example for PRA
   - An example for PA
   - An abstract approach

OUTLINE

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Figure: A cherry tree in Kenrokuen garden.

Figure: A drawing of Kenrokuen garden.

In September 2018 there will be a summer school and a workshop on proof theory in Ghent. Interested (master, PhD) students or other interested people are welcome.

MOTIVATION

Gödel: $(\exists A \in L_{\mathrm{PA}})[\mathbb{N} \models A \;\&\; \mathrm{PA} \nvdash A]$

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Motivation

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Problem: Find natural examples for A
One possible solution: Goodstein principles
Challenge: What do Goodstein principles say about natural well orderings?

A simple example for PRA

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$$m =_{k-NF} k \cdot p + q.$$

If $m < k$ then $m[k \leftarrow k+1] := m$. For $m =_{k-NF} k \cdot p + q$ define recursively

$$m[k \leftarrow k+1] := (k+1) \cdot p[k \leftarrow k+1] + q.$$

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$$m_{k+1} := m_k[k \leftarrow k+1] - 1$$

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4. $\psi_k m = \psi_{k+1} m[k \leftarrow k+1]$. (Property C4)
5. If $m < n$ then $\psi_k m < \psi_k n$. (Property C5)

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Let $ord(m,l) := \psi_{l+2}(m_l)$. If $m_l > 0$ then

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This yields a primitive recursive descending chain of ordinals which cannot be infinitely long.

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1. If $m > 0$ then $\psi_{k+1}(m[k \leftarrow k+1] - 1) \geq (\psi_k m)[k-1]$.
2. $\alpha(m, l) \leq o(2m, l)$.

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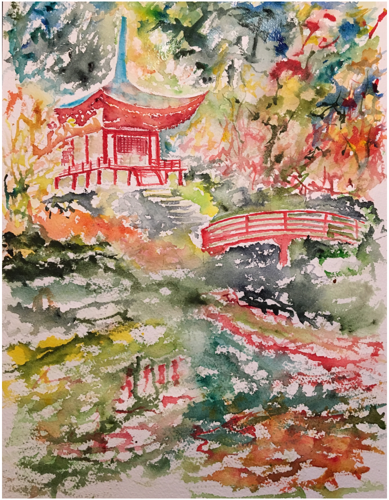

Figure: A temple in Kyoto.

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Figure: The gate of Myajima.

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Figure: The atomic bomb ruin in Hiroshima.

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The normal forms produce terms of minimal lengths. Moreover they become maximal under base change.

NORM MINIMALITY

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Let $m \geq 1$. If $m < k$ then $Nm := m$. If $m =_{NF} k \cdot p + q$ where $q < k$ and $p > 0$ then $Nm := 1 + Np + q$.

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Lemma

1. $N(m+1) \leq Nm + 1$.
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3. If $m = k \cdot l + n$ not necessarily a normal form then $Nm \leq 1 + Nl + n$.

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2. If $l \in T_k$ and $n < \omega$ then $k \cdot l + n \in T_k$ and $M(k \cdot l + n) := 1 + M(l) + n$ and $val(t) = k \cdot val(l) + n$.

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Theorem

If $t \in T_k$ and $val(t) = m$ then $Nm = M(t(m)) \leq Mt$. So among all representations of m the normal form representation has the smallest M norm. (Property C6)

Recap: $m[k \leftarrow k+1] := m$ if $m < k$ and that $m[k \leftarrow k+1] := (k+1) \cdot p[k \leftarrow k+1] + q$ if $m =_{k-NF} k \cdot p + q$ where $q < k$ and $p > 0$.

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If $m = k \cdot p' + q'$ (not necessarily in normal form) then $m[k \leftarrow k+1] \geq (k+1) \cdot p'[k \leftarrow k+1] + q'$.

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Figure: A flower.

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Our normal form satisfies (C1)-(C5).

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If $m = A_p(k, q)$ then
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Figure: An impression from Scheldevelde.

EXTENSION TO ATR_0

To obtain an independence result for ATR_0 we just have to define $A_0(k,b) := k^b$ instead of $A_0(k,b) := b+1$. Then (C1)-(C5) and (C7) hold and one obtains a sharp independence result for ATR_0.

Figure: The open sea.

Extension to the Schwichtenberg Wainer hierarchy

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We assume given a canonical assignment of fundamental sequences for the limits below ε_0.
Let $A_0(k, b) := b + 1$.
Let $A_{\alpha+1}(k, 0) := A_\alpha^k(k, \cdot)(1)$ and
$A_{\alpha+1}(k, b + 1) := A_\alpha^k(k, \cdot)(A_{\alpha+1}(k, b))$.

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If α is a limit then put $A_\alpha(k, 0) := A_{\alpha_k}(k, 0)$ where $\alpha_0 = \alpha[0]$ and $\alpha_{k+1} = \alpha[A_{\alpha_k}(k, 0)]$. Moreover put $A_\alpha(k, b + 1) := A_{\alpha_k}(k, 0)$ where $\alpha_0 = \alpha[A_\alpha(k, b)]$ and $\alpha_{k+1} = \alpha[A_{\alpha_k}(k, 0)]$.

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$$\begin{aligned}\alpha[k \leftarrow k+1] &= \omega^{\alpha_1[k\leftarrow k+1]} \cdot m_1[k \leftarrow k+1] \\ &\quad + \cdots + \omega^{\alpha_k[k\leftarrow k+1]} \cdot m_k[k \leftarrow k+1].\end{aligned}$$

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We believe being able to prove (C7) and that $(\Pi^1_1 - \mathrm{CA})^-_0 \nvdash (\forall l)(\exists k) m_k = 0)$. Moreover we want to go far beyond η_0. This might confront us with a rough sea of complications. A longterm goal will be to classify the second recursively inaccessible ordinal which we conjecture being equal to $\psi_0(\Omega_{\Omega_\omega})$.

Arigato. (Thank you for listening.)
The basics of sub recursion theory will be covered in the September summer school.

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