On a computational interpretation of sequent calculus for modal logic S4

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Some studies [Kobayashi '97][Benton+ '98][Pfenning+ '01][Kimura+ '11] discovered that S4 corresponds to various typed λ-calculi for “meta-programming”

In the logical foundation, □-modality plays an essential role:

- □A means the set of programs which “encode” programs of type A
- (this is similar to the intuition in Logic of Proof, etc: □A means the proposition of a “proof” of A)
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In the logical foundation, $\Box$-modality plays an essential role:

- $\Box A$ means the set of programs which “encode” programs of type $A$
- (this is similar to the intuition in Logic of Proof, etc: $\Box A$ means the proposition of a “proof” of $A$)
All the previous studies only consider natural-deduction-style $\lambda$-calculi, and they use the “one-step” substitution as usual:

$$(\lambda x. M) N \rightsquigarrow M[x := N]$$

However, the operation is too rich from a practical viewpoint.

Natural-deduction-style $\lambda$-calculus is not enough to capture the structure of computation.
Problem from a practical viewpoint

All the previous studies only consider natural-deduction-style \( \lambda \)-calculi, and they use the “one-step” substitution as usual:

\[
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\]

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This talk

Aim of this talk

To create another computational model for modal logic S4, in terms of sequent calculus

To do this, a sequent calculus and its corresponding calculus for intuitionistic S4 are proposed

1 proof-theoretically based on:
   - a modal sequent calculus and the G3-style system [Troelstra&Schwichtenberg '96]
   - a higher-arity modal natural deduction [Pfenning&Davies '01]

2 type-theoretically based on:
   - the higher-arity modal λ-calculus [Pfenning&Davies '01]
   - the Curry–Howard correspondence for a G3-style sequent calc. [Ohori '99]
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Higher-arity Sequent Calculus for intuitionistic S4

We propose a “higher-arity” sequent calc. for \((\land, \lor, \supset, \Box)-\text{fragment}\) of intuitionistic S4, \(\text{HLJ}_{S4}\), based on [Troelstra&Schwichtenberg ’96]

Definition (Formula)

\[
A, B ::= p \mid A \land B \mid A \lor B \mid A \supset B \mid \Box A
\]

Definition (Higher-arity judgment [Pfenning+ ’01])

A \textit{judgment} is defined by the following higher-arity form:

\[
\Delta; \Gamma \vdash A
\]

which intuitively means \((\land \Box \Delta) \land (\land \Gamma) \supset A\)
Inference rules of $\text{HLJ}_{S4}$

$\text{Ax}$

\[
\begin{align*}
\Delta; \Gamma \vdash A & \quad \Delta; \Gamma \vdash B \\
\Delta; \Gamma \vdash A \land B
\end{align*}
\]

$\land R$

\[
\begin{align*}
\Delta; \Gamma \vdash A_i \\
\Delta; \Gamma \vdash A_1 \lor A_2 \\
\Delta; \Gamma \vdash B \\
\Delta; \Gamma \vdash A \lor B
\end{align*}
\]

$\lor R$

\[
\begin{align*}
\Delta; \emptyset \vdash A \\
\Delta; \emptyset \vdash A
\end{align*}
\]

$\Box R$

\[
\begin{align*}
\Delta; \Gamma \vdash B \\
\Delta; \Gamma \vdash B
\end{align*}
\]

$\text{W}_\Box$

\[
\begin{align*}
\Delta; \Gamma \vdash A, A, A \vdash B \\
\Delta; \Gamma, A, A \vdash B
\end{align*}
\]

$\text{C}$

\[
\begin{align*}
\Delta; \Gamma \vdash A, \Delta; \Gamma \vdash B \\
\Delta; \Gamma \vdash B
\end{align*}
\]

Cut

\[
\begin{align*}
\Delta; \emptyset \vdash A & \quad \Delta, A; \Gamma \vdash B \\
\Delta; \Gamma \vdash B
\end{align*}
\]

$\Box \text{Cut}$
Init rules of $\mathbf{HLJ}_{S4}$

Init rule

\[
\emptyset; A \vdash A \quad \text{Ax}
\]

\[
 A; \emptyset \vdash A \quad \Box \text{Ax}
\]

Intuition of $\text{Ax}$

\[
\Box A \supset A \quad \Box \text{Ax}
\]
Init rules of $\text{HLJ}_{S4}$

Init rule

\[ \emptyset; A \vdash A \quad \text{Ax} \]

\[ A; \emptyset \vdash A \quad \Box \text{Ax} \]

Intuition of $\text{Ax}$

\[ \Box A \supset A \quad \Box \text{Ax} \]
Logical rules of $\mathbf{HLJ}_{S4}$

**Logical rule**

\[
\begin{align*}
  & \Delta; \Gamma \vdash A & \Delta; \Gamma \vdash B \\
  \quad & \frac{}{\Delta; \Gamma \vdash A \land B} & \land R \\
  & \Delta; \Gamma \vdash A_i \\
  \quad & \frac{}{\Delta; \Gamma \vdash A_1 \lor A_2} & \lor R \\
  & \Delta; \Gamma, A \vdash B \\
  \quad & \frac{}{\Delta; \Gamma \vdash A \supset B} & \supset R \\
  & \Delta; \emptyset \vdash A \\
  \quad & \frac{}{\Delta; \emptyset \vdash \Box A} & \Box R \\
  & \Delta; \Gamma, \emptyset \vdash A \\
  \quad & \frac{}{\Box \Gamma, \square A \vdash B} & \Box L
\end{align*}
\]

\[
\begin{align*}
  & \Delta; \Gamma, A_1 \land A_2 \vdash B \\
  \quad & \frac{}{\Delta; \Gamma \vdash A \land B} & \land L \\
  & \Delta; \Gamma, A \vdash C \\
  \quad & \frac{}{\Delta; \Gamma \vdash A \lor B \vdash C} & \lor L \\
  & \Delta; \Gamma, A \vdash B \\
  \quad & \frac{}{\Delta; \Gamma \vdash A \supset B \vdash C} & \supset L \\
  & \Delta, A; \Gamma \vdash B \\
  \quad & \frac{}{\Delta; \Gamma \vdash \Box A \vdash B} & \Box L
\end{align*}
\]
Logical rules of \( \text{HLJ}_{\text{s4}} \)

**Logical rule**

\[
\frac{\Delta; \Gamma \vdash A}{\Delta; \Gamma \vdash A \land B} \quad \land R
\]

\[
\frac{\Delta; \Gamma \vdash B}{\Delta; \Gamma \vdash A \land B}
\]

\[
\frac{\Delta; \Gamma \vdash A}{\Delta; \Gamma \vdash A \lor B} \quad \lor R
\]

\[
\frac{\Delta; \Gamma \vdash A \lor B}{\Delta; \Gamma \vdash A \lor B}
\]

\[
\frac{\Delta; \Gamma, A_i \vdash B}{\Delta; \Gamma, A_1 \land A_2 \vdash B} \quad \land L
\]

\[
\frac{\Delta; \Gamma, A_1 \land A_2 \vdash B}{\Delta; \Gamma, A_i \vdash B}
\]

\[
\frac{\Delta; \Gamma \vdash A}{\Delta; \Gamma \vdash A \lor B} \quad \lor L
\]

\[
\frac{\Delta; \Gamma \vdash A \lor B}{\Delta; \Gamma \vdash A \lor B}
\]

\[
\frac{\Delta; \Gamma \vdash A \lor B}{\Delta; \Gamma, A \lor B \vdash C} \quad \lor L
\]

\[
\frac{\Delta; \Gamma \vdash A \lor B}{\Delta; \Gamma, A \lor B \vdash C}
\]

\[
\frac{\Delta; \Gamma \vdash A \lor B}{\Delta; \Gamma, A \lor B \vdash C} \quad \lor L
\]

\[
\frac{\Delta; \Gamma, A \lor B \vdash C}{\Delta; \Gamma \vdash A \lor B}
\]

\[
\frac{\Delta; \Gamma \vdash A \lor B}{\Delta; \Gamma \vdash A \lor B}
\]

\[
\frac{\Delta; \Gamma \vdash A \lor B}{\Delta; \Gamma \vdash A \lor B}
\]

\[
\frac{\Delta; \Gamma \vdash A \lor B}{\Delta; \Gamma \vdash A \lor B}
\]

\[
\frac{\Delta; \Gamma \vdash A \lor B}{\Delta; \Gamma \vdash A \lor B}
\]

\[
\frac{\Delta; \Gamma \vdash A \lor B}{\Delta; \Gamma \vdash A \lor B}
\]
Logical rules of $\text{HLJ}_{S_4}$

**Logical rule**

$\Delta; \Gamma \vdash A$  $\Delta; \Gamma \vdash B$

$\Delta; \Gamma \vdash A \land B$

$\Delta; \Gamma \vdash A_i$

$\Delta; \Gamma \vdash A \lor A_i$

$\Delta; \Gamma, A \vdash B$

$\Delta; \Gamma \vdash A \lor B$

$\Delta; \Gamma \vdash A \\
\Delta; \Gamma \vdash B$

$\Delta; \Gamma \vdash A \supset B$

$\Delta; \Gamma, A \vdash A_i$

$\Delta; \Gamma, A_i \vdash B$

$\Delta; \Gamma \vdash A_i \land B$

$\Delta; \Gamma \vdash A \lor B$

$\Delta; \Gamma, A \lor B \vdash C$

$\Delta; \Gamma \vdash A$  $\Delta; \Gamma \vdash C$

$\Delta; \Gamma, A \lor B \vdash C$

$\Delta; \Gamma \vdash A$  $\Delta; \Gamma \vdash B$

$\Delta; \Gamma \vdash A \supset B$

$\Delta; \Gamma, A \supset B \vdash C$

$\Delta; \Gamma \vdash A \supset B$

$\Delta; \Gamma \vdash A \land B$

$\Delta; \Gamma, A \land B \vdash C$

$\Delta; \Gamma \vdash A \land B$

$\Delta; \Gamma, A \land B \vdash C$

$\Delta, \Gamma \vdash B$

$\Delta; \Gamma, A \vdash B$

$\Delta; \Gamma, A \vdash C$

$\Delta; \Gamma, A \supset B \vdash C$

$\Delta; \Gamma \vdash A \land B$

$\Delta; \Gamma, A \land B \vdash C$

$\Delta; \Gamma \vdash A$  $\Delta; \Gamma \vdash B$

$\Delta; \Gamma \vdash A \lor B$

$\Delta; \Gamma, A \lor B \vdash C$

$\Delta; \Gamma \vdash A \lor B$

$\Delta; \Gamma, A \lor B \vdash C$

$\Delta; \Gamma \vdash A \lor B$

$\Delta; \Gamma, A \lor B \vdash C$

$\Delta; \Gamma \vdash A \lor B$

$\Delta; \Gamma, A \lor B \vdash C$
Structural rules of $\text{HLJ}_{S4}$

**Structural rule**

$$
\frac{\Delta; \Gamma \vdash B}{\Delta; \Gamma, A \vdash B} \quad \text{W}
$$

$$
\frac{\Delta; \Gamma, A, A \vdash B}{\Delta; \Gamma, A \vdash B} \quad \text{C}
$$

**Cut rule**

$$
\frac{\Delta; \Gamma \vdash A \quad \Delta; \Gamma, A \vdash B}{\Delta; \Gamma \vdash B} \quad \text{Cut}
$$

$$
\frac{\Delta; \emptyset \vdash A \quad \Delta, A; \Gamma \vdash B}{\Delta; \Gamma \vdash B} \quad \Box \text{Cut}
$$

$$
\frac{\Delta, A, A; \Gamma \vdash B}{\Delta, A; \Gamma \vdash B} \quad \Box \text{C}
$$
On the cut-elimination procedure

While we can prove the cut-elimination theorem for $\textbf{HLJ}_{S4}$, the proof by the mix-elimination is problematic; because ...

\[
\begin{array}{c}
\Pi \\
\Delta; \Gamma \vdash A \\
\hline
\Pi' \\
\Delta'; \Gamma', A, A \vdash B \\
\end{array}
\quad \Rightarrow \quad
\begin{array}{c}
\Pi \\
\Delta; \Gamma \vdash A \\
\Delta'; \Gamma', A, A \vdash B \\
\hline
\end{array}
\quad \text{Cut}
\quad
\begin{array}{c}
\Pi \\
\Delta; \Gamma \vdash A \\
\Delta'; \Gamma', A, A \vdash B \\
\hline
\end{array}
\quad \text{Mix}
\]

In the elimination procedure,

- it is “okay” if we consider the provability of the judgment; but
- it is “not okay” if we consider the construction of the judgment
While we can prove the cut-elimination theorem for $\text{HLJ}_{S4}$, the proof by the mix-elimination is problematic; because ...

\[
\frac{\Pi'}{\Pi} \quad \frac{\Delta; \Gamma \vdash A}{\Delta'; \Gamma' \vdash A, A \vdash B} \quad \frac{\Delta'; \Gamma', A, A \vdash B}{\Delta, \Delta'; \Gamma, \Gamma' \vdash B} \quad \text{Cut} \quad \frac{\Pi'}{\Pi} \quad \frac{\Delta; \Gamma \vdash A}{\Delta'; \Gamma' \vdash A, A \vdash B} \quad \frac{\Delta, \Delta'; \Gamma, \Gamma' \vdash B}{\Delta, \Delta'; \Gamma, \Gamma' \vdash B} \quad \text{Mix}
\]

In the elimination procedure,

- it is “okay” if we consider the provability of the judgment; but
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While we can prove the cut-elimination theorem for $\text{HLJ}_{S4}$, the proof by the mix-elimination is problematic; because ...

\[
\frac{\Pi'}{\Delta; \Gamma \vdash A} \quad \frac{\Delta'; \Gamma', \Delta; \Gamma \vdash B}{\Pi} \quad \frac{\Delta'; \Gamma', \Delta; \Gamma \vdash B}{\Pi'} \quad \frac{\Pi}{\Delta; \Gamma \vdash A} \quad \frac{\Delta'; \Gamma', \Delta; \Gamma \vdash B}{\Pi'}
\]

In the elimination procedure,

- it is “okay” if we consider the provability of the judgment; but
- it is “not okay” if we consider the construction of the judgment
The *G3-style* [Kleene ’52][Dragalin ’88] is a style of formalization to make a cut-free, or precisely, “structural-rule-free” system.

The G3-style inference rules are defined in a somewhat tricky way to derive the “height-preserving admissible” structural rules.
G3-style system for $\text{HLJ}_{S4}$, named $\text{G3-HLJ}_{S4}$

The G3-style inference rules are defined as follows:

- **Ax**
  \[ \frac{}{\Delta; \Gamma, A \vdash A} \]

- **□Ax**
  \[ \frac{}{\Delta, A; \Gamma \vdash A} \]

- **⊃R**
  \[ \frac{\Delta; \Gamma, A \vdash B}{\Delta; \Gamma \vdash A \supset B} \]

- **⊃L**
  \[ \frac{\Delta; \Gamma, A \vdash B}{\Delta; \Gamma \vdash A \supset B} \]

- **∧R**
  \[ \frac{\Delta; \Gamma \vdash A \quad \Delta; \Gamma \vdash B}{\Delta; \Gamma \vdash A \land B} \]

- **∧L**
  \[ \frac{\Delta; \Gamma, A \land B, A, B \vdash C}{\Delta; \Gamma, A \land B \vdash C} \]

- **∨R**
  \[ \frac{\Delta; \Gamma \vdash A_i}{\Delta; \Gamma \vdash A_1 \lor A_2} \]

- **∨L**
  \[ \frac{\Delta; \Gamma, A \lor B, A \vdash C \quad \Delta; \Gamma, A \lor B, B \vdash C}{\Delta; \Gamma, A \lor B \vdash C} \]

- **□R**
  \[ \frac{\Delta; \emptyset \vdash A}{\Delta; \Gamma \vdash \Box A} \]

- **□L**
  \[ \frac{\Delta, A; \Gamma, \Box A \vdash B}{\Delta; \Gamma, \Box A \vdash B} \]
From the original rules to the G3-style

Idea: all we have to do is to get “height-preserving” structural rules

Original $\text{HLJ}_{S4}$

- $\emptyset; A \vdash A$ \hspace{1cm} $\text{Ax}$
- $\Delta; \Gamma, A_i \vdash B$ \hspace{1cm} $\Delta; \Gamma, A_1 \land A_2 \vdash B$ \hspace{1cm} $\land L$
- $\Delta, A; \Gamma \vdash B$ \hspace{1cm} $\Delta; \Gamma, \Box A \vdash B$ \hspace{1cm} $\Box L$

G3-style $\text{G3-HLJ}_{S4}$

- $\Delta; \Gamma, A \vdash A$ \hspace{1cm} $\text{Ax}$
- $\Delta; \Gamma, A \land B, A, B \vdash C$ \hspace{1cm} $\Delta; \Gamma, A \land B \vdash C$ \hspace{1cm} $\land L$
- $\Delta, A; \Gamma, \Box A \vdash B$ \hspace{1cm} $\Delta; \Gamma, \Box A \vdash B$ \hspace{1cm} $\Box L$
Desired properties

**Lemma (Height-preserving weakening/contraction)**

The followings are height-preserving admissible rules in $\text{G3-\text{HLJ}}_{S4}$:

\[
\frac{\Delta; \Gamma \vdash B}{\Delta; \Gamma, A \vdash B} \quad W
\]

\[
\frac{\Delta; \Gamma, A, A \vdash B}{\Delta; \Gamma \vdash B} \quad C
\]

\[
\frac{\Delta; \Gamma \vdash B}{\Delta, A; \Gamma \vdash B} \quad \Box W
\]

\[
\frac{\Delta, A, A; \Gamma \vdash B}{\Delta, A; \Gamma \vdash B} \quad \Box C
\]

**Theorem (Equivalence)**

The provability of $\text{HLJ}_{S4}$ and $\text{G3-\text{HLJ}}_{S4} + \text{Cut}$ is equivalent.

**Theorem (Cut-elimination)**

The cut rules $\text{Cut}$ and $\Box \text{Cut}$ are admissible in $\text{G3-\text{HLJ}}_{S4}$.
Desired properties

**Lemma (Height-preserving weakening/contraction)**

The followings are height-preserving admissible rules in $\text{G}_3\text{-HLJ}_{S_4}$:

- $\Delta; \Gamma \vdash B \quad W$
- $\Delta; \Gamma, A \vdash B \quad C$
- $\Delta; \Gamma \vdash B \\ ^{\square}W$
- $\Delta, A; \Gamma \vdash B \\ ^{\square}C$

**Theorem (Equivalence)**

The provability of $\text{HLJ}_{S_4}$ and $\text{G}_3\text{-HLJ}_{S_4} + \text{Cut}$ is equivalent.

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Desired properties

Lemma (Height-preserving weakening/contraction)

The followings are height-preserving admissible rules in \( \text{G3-HLJ}_{S4} \):

\[
\begin{align*}
\Delta; \Gamma \vdash B & \quad \text{W} \\
\Delta; \Gamma, A \vdash B & \\
\Delta; \Gamma, A, A \vdash B & \quad \text{C} \\
\Delta; \Gamma, A, A \vdash B & \\
\Delta; \Gamma \vdash B & \quad \Box W \\
\Delta, A; \Gamma \vdash B & \\
\Delta, A, A; \Gamma \vdash B & \quad \Box C \\
\Delta, A; \Gamma \vdash B & \\
\Delta, A \vdash B & \\
\Delta \vdash B & \\
\Delta; \Gamma \vdash B & \\
\Delta, A; \Gamma \vdash B & \quad \Box C
\end{align*}
\]

Theorem (Equivalence)

The provability of \( \text{HLJ}_{S4} \) and \( \text{G3-HLJ}_{S4} + \text{Cut} \) is equivalent

Theorem (Cut-elimination)

The cut rules \( \text{Cut} \) and \( \Box \text{Cut} \) are admissible in \( \text{G3-HLJ}_{S4} \)
We propose a term assignment system for the $\text{G3-}\text{HLJ}_{S4}$, $\lambda_{seq}$, to get the computational model

As [Ohori ’99] did for a G3-style prop. int. sequent calc., we assign terms to $\text{G3-}\text{HLJ}_{S4} + \text{Cut}$ as follows:

- **Init/Right rules**: assign $\lambda$-terms, as we do for N.D. system
- **Left/Cut rules**: assign the so-called “let expression”

Good point: $\lambda_{seq}$ does not use “meta-level” substitution!
We propose a term assignment system for the $\text{G3-HLJ}_{\mathbf{S}_4}$, $\lambda_{\Box_{\text{seq}}}$, to get the computational model

As [Ohori ’99] did for a G3-style prop. int. sequent calc., we assign terms to $\text{G3-HLJ}_{\mathbf{S}_4} + \text{Cut}$ as follows:

- Init/Right rules: assign $\lambda$-terms, as we do for N.D. system
- Left/Cut rules: assign the so-called “let expression”

Good point: $\lambda_{\Box_{\text{seq}}}$ does not use “meta-level” substitution!
Assign the modal $\lambda$-term [Pfenning+ ’01] to the init/right rules:

\[
\begin{align*}
\Delta; \Gamma, x : A & \vdash x : A \quad \text{Ax} & \\
\Delta, u : A; \Gamma & \vdash u : A \quad \Box \text{Ax} \\
\Delta; \Gamma & \vdash M : A & \\
\Delta; \Gamma & \vdash N : B & \\
\Delta; \Gamma & \vdash \langle M, N \rangle : A \land B \quad \land R \\
\Delta; \Gamma & \vdash \lambda x : A. M : B & \\
\Delta; \Gamma & \vdash M : A \supset B \quad \supset R \\
\Delta; \emptyset & \vdash M : A & \\
\Delta; \Gamma & \vdash \text{box} M : \Box A \quad \Box R
\end{align*}
\]
Assign “let-expression” to the left conjunction rule:

\[
\frac{\Delta; \Gamma, x : A \land B, y : A, z : B \vdash M : C}{\Delta; \Gamma, x : A \land B \vdash \text{let } \langle y, z \rangle = x \text{ in } M : C} \quad ^{\land \text{L}}
\]

The reduction intuitively proceeds, e.g., as:

\[
(\text{let } \langle y, z \rangle = \langle N, L \rangle \text{ in } M) \rightsquigarrow M[y := N, z := L]
\]
Assign “let-expression” to the left conjunction rule:

\[
\frac{\Delta; \Gamma, x : A \land B, y : A, z : B \vdash M : C}{\Delta; \Gamma, x : A \land B \vdash \text{let } \langle y, z \rangle = x \text{ in } M : C} \quad \land L
\]

The reduction intuitively proceeds, e.g., as:

\((\text{let } \langle y, z \rangle = \langle N, L \rangle \text{ in } M) \rightsquigarrow M[y := N, z := L]\)
Assign “let-expression” to the left conjunction rule:

\[
\frac{\Delta; \Gamma, x : A \land B, y : A, z : B \vdash M : C}{\Delta; \Gamma, x : A \land B \vdash \text{let} \langle y, z \rangle = x \text{ in } M : C} \quad \land L
\]

The reduction intuitively proceeds, e.g., as:

\[
(\text{let} \langle y, z \rangle = \langle N, L \rangle \text{ in } M) \leadsto M[y := N, z := L]
\]
Assign “let-expression” to the left conjunction rule:

\[
\frac{
  \Delta; \Gamma, x : A \land B, y : A, z : B \vdash M : C
}{
  \Delta; \Gamma, x : A \land B \vdash \text{let} \langle y, z \rangle = x \text{ in } M : C
}\]

\(\land L\)

The reduction intuitively proceeds, e.g., as:

\((\text{let} \langle y, z \rangle = \langle N, L \rangle \text{ in } M) \leadsto M[y := N, z := L]\)
The rules for the other left rules are defined similarly:

\[
\begin{align*}
\Delta; \Gamma, x : A \supset B & \vdash M : A & & \Delta; \Gamma, x : A \supset B, y : B & \vdash N : C \\
\Delta; \Gamma, x : A \supset B & \vdash \text{let } y = x \text{ in } N : C & & \text{\textbf{\(\square L\)}}
\end{align*}
\]

\[
\begin{align*}
\Delta, u : A; \Gamma, x : \Box A & \vdash M : B & & \Delta; \Gamma, x : \Box A & \vdash \text{let box } u = x \text{ in } M : B & & \Box L
\end{align*}
\]
The term assignment for cut rules are defined as a “composition” of two constructions, again by using let-expressions:

\[
\frac{\Delta; \Gamma \vdash M : A \quad \Delta; \Gamma, x : A \vdash N : B}{\Delta; \Gamma \vdash \text{let } x = M \text{ in } N : B} \quad \text{Cut}
\]

\[
\frac{\Delta; \emptyset \vdash M : A \quad \Delta, u : A; \Gamma \vdash N : B}{\Delta; \Gamma \vdash \text{let } u = M \text{ in } N : B} \quad \Box \text{Cut}
\]
Let us consider the cut-elimination for conjunction:

\[
\frac{\vdash M : A \quad \vdash N : B}{\vdash \langle M, N \rangle : A \land B} \quad \text{\(\land R\)}
\]
\[
\frac{x : A \land B, y : A, z : B \vdash L : C}{x : A \land B \vdash \text{let } \langle y, z \rangle = x \text{ in } L : C} \quad \text{\(\land L\)}
\]

\[
\frac{\vdash \text{let } x = \langle M, N \rangle \text{ in let } \langle y, z \rangle = x \text{ in } L : C}{\vdash \text{let } x = \langle M, N \rangle \text{ in let } \langle y, z \rangle = x \text{ in } L : C} \quad \text{Cut}
\]

To eliminate cuts, all we have to do is to compute:

\[
(\text{let } x = \langle M, N \rangle \text{ in let } \langle y, z \rangle = x \text{ in } L)
\]

\[
\leadsto L[y := M, z := N, x := \langle M, N \rangle]
\]

but we do not want to use “meta-level” substitution.

Fortunately, the (local) cut-elimination step defined in the G3-style is exactly what we want!
Cut-elimination in terms of $\lambda_{seq}$

Let us consider the cut-elimination for conjunction:

$$
\frac{
\vdash M : A \quad \vdash N : B
}{
\vdash \langle M, N \rangle : A \land B}
\quad
\frac{
x : A \land B, y : A, z : B \vdash L : C
}{
\vdash \text{let } \langle y, z \rangle = x \text{ in } L : C}
\quad
\frac{
x : A \land B \vdash L : C
}{
\text{Cut}}
$$

To eliminate cuts, all we have to do is to compute:

$$(\text{let } x = \langle M, N \rangle \text{ in let } \langle y, z \rangle = x \text{ in } L)$$

$$\leadsto L[y := M, z := N, x := \langle M, N \rangle]$$

but we do not want to use “meta-level” substitution

Fortunately, the (local) cut-elimination step defined in the G3-style is exactly what we want!
Let us consider the cut-elimination for conjunction:

\[
\frac{\vdash M : A \quad \vdash N : B}{\vdash \langle M, N \rangle : A \land B} \quad \land R
\]

\[
\frac{x : A \land B, y : A, z : B \vdash L : C}{\vdash \text{let } \langle y, z \rangle = x \text{ in } L : C} \quad \land L
\]

\[
\frac{\vdash \text{let } x = \langle M, N \rangle \text{ in let } \langle y, z \rangle = x \text{ in } L : C}{\vdash \text{let } x = \langle M, N \rangle \text{ in let } \langle y, z \rangle = x \text{ in } L : C} \quad \text{Cut}
\]

To eliminate cuts, all we have to do is to compute:

\[
(\text{let } x = \langle M, N \rangle \text{ in let } \langle y, z \rangle = x \text{ in } L)
\]

\[
\leadsto L[y := M, z := N, x := \langle M, N \rangle]
\]

but we do not want to use “meta-level” substitution

Fortunately, the (local) cut-elimination step defined in the G3-style is exactly what we want!
Local cut-elimination as program-simplification

(A part of) translation rules are obtained as follows:

**Optimization**

\[(\text{let } x = M \text{ in } x) \rightsquigarrow M\]
\[(\text{let } x = M \text{ in } y) \rightsquigarrow y\]

**Flattening**

\[(\text{let } w = (\text{let } \langle y, z \rangle = x \text{ in } M) \text{ in } N) \rightsquigarrow (\text{let } \langle y, z \rangle = x \text{ in } \text{let } w = M \text{ in } N)\]
\[(\text{let } y = (\text{let } \text{box } u = x \text{ in } M) \text{ in } N) \rightsquigarrow (\text{let } \text{box } u = x \text{ in } \text{let } y = M \text{ in } N)\]

**Decomposition**

\[(\text{let } x = \langle M, N \rangle \text{ in } \text{let } \langle y, z \rangle = x \text{ in } L) \rightsquigarrow (\text{let } y = M \text{ in } \text{let } z = N \text{ in } \text{let } x = \langle y, z \rangle \text{ in } L)\]
\[(\text{let } x = \text{box } M \text{ in } \text{let } \text{box } u = x \text{ in } N) \rightsquigarrow (\text{let } u = M \text{ in } \text{let } x = \text{box } u \text{ in } N)\]

These translation corresponds to “A-normal form compilation” in the theory of programming languages.
(A part of) translation rules are obtained as follows:

**Optimization**

(\texttt{let } x = M \texttt{ in } x) \rightsquigarrow M \\
(\texttt{let } x = M \texttt{ in } y) \rightsquigarrow y

**Flattening**

(\texttt{let } w = \langle \texttt{let } y, z \rangle = x \texttt{ in } M \rangle \texttt{ in } N) \rightsquigarrow (\texttt{let } \langle y, z \rangle = x \texttt{ in } \texttt{let } w = M \texttt{ in } N) \\
(\texttt{let } y = \langle \texttt{let } box u = x \texttt{ in } M \rangle \texttt{ in } N) \rightsquigarrow (\texttt{let } box u = x \texttt{ in } \texttt{let } y = M \texttt{ in } N)

**Decomposition**

(\texttt{let } x = \langle M, N \rangle \texttt{ in } \texttt{let } \langle y, z \rangle = x \texttt{ in } L) \rightsquigarrow (\texttt{let } y = M \texttt{ in } \texttt{let } z = N \texttt{ in } \texttt{let } x = \langle y, z \rangle \texttt{ in } L) \\
(\texttt{let } x = box M \texttt{ in } \texttt{let } box u = x \texttt{ in } N) \rightsquigarrow (\texttt{let } u = M \texttt{ in } \texttt{let } x = box u \texttt{ in } N)

These translation corresponds to “A-normal form compilation” in the theory of programming languages.
Properties of $\lambda_{seq}$ and the cut-elimination theorem

**Theorem (Subject reduction)**

If $\Delta; \Gamma \vdash M : A$ and $M \leadsto M'$, then $\Delta; \Gamma \vdash M' : A$

**Theorem (Strong normalization)**

*Every typable term is strongly normalizing*

**Corollary (Cut-elimination theorem)**

$\lambda_{seq}$ enjoys the cut-elimination theorem, which also yields that every typable term can be reduced to the unique normal form
The following tells us that $\lambda^{\square}_{\text{seq}}$ can be used as a basis of model for the existing theory:

**Theorem (Embedding from modal typed $\lambda$-calculus)**

The modal $\lambda$-calc. $\lambda^{\square}$ [Pfenning+ ’01] can be embeded into $\lambda^{\square}_{\text{seq}}$:

- If $\Delta; \Gamma \vdash M : A$ in $\lambda^{\square}$, then $\Delta; \Gamma \vdash \langle M \rangle : A$ in $\lambda^{\square}_{\text{seq}}$
- If $M \rightsquigarrow M'$ in $\lambda^{\square}$, then $\langle M \rangle \rightsquigarrow \langle M' \rangle$ in $\lambda^{\square}_{\text{seq}}$

where $\langle \square \rangle$ means the translation mapping from $\lambda^{\square}$ to $\lambda^{\square}_{\text{seq}}$
Conclusion and future work

- **Conclusion**
  - A cut-free higher-arity sequent calc. for intuitionistic S4: $\text{HLJ}_{S4}$ and $\text{G3-HLJ}_{S4}$
  - (A cut-free higher-arity sequent calc. for classical S4: $\text{HLK}_{S4}$ and $\text{G3-HLK}_{S4}$)
  - The corresponding term calculus for $\text{G3-HLJ}_{S4}$

- **Future work**
  - The corresponding term calculus for the classical version, following the work of $\lambda\mu$-calculus for modal logic [Kimura+ ’11]
  - (Ongoing work with Akira Yoshimizu): Geometry of Interaction semantics for modal logic in terms of MELL, following the work of GoI semantics for PCF [Mackie ’95]
let \( x = y \) in \( M \) \( \leadsto \) \( M[x := y] \)

let \( x = u \) in \( M \) \( \leadsto \) \( M[x := u] \)

let \( u = v \) in \( M \) \( \leadsto \) \( M[u := v] \)

let \( x = M \) in \( x \) \( \leadsto \) \( M \)

let \( x = M \) in \( y \) \( \leadsto \) \( y \)

let \( u = M \) in \( x \) \( \leadsto \) \( x \)

let \( u = M \) in \( u \) \( \leadsto \) \( M \)

let \( u = M \) in \( v \) \( \leadsto \) \( v \)

let \( x = M \) in \( u \) \( \leadsto \) \( u \)

let \( z = (\text{let } y = x \text{ in } M) \text{ in } N \) \( \leadsto \) let \( y = x \text{ in } M \text{ in } z = N \text{ in } L \)

let \( w = (\text{let } \langle y, z \rangle = x \text{ in } M) \text{ in } N \) \( \leadsto \) let \( \langle y, z \rangle = x \text{ in } \text{ let } w = M \text{ in } N \)

let \( w = (\text{case } x \text{ of } [y]M \text{ or } [z]N) \text{ in } L \) \( \leadsto \) case \( x \text{ of } [y](\text{let } w = M \text{ in } L) \text{ or } [z](\text{let } w = N \text{ in } L) \)

let \( y = (\text{let box } u = x \text{ in } M) \text{ in } N \) \( \leadsto \) let \( \text{box } u = x \text{ in } \text{ let } y = M \text{ in } N \)
let $x = L$ in let $z = y \; M$ in $N \leadsto$ let $z = y \; (\text{let } x = L \; \text{in } M)$ in let $x = L \; \text{in } N$

let $x = N$ in let $\langle y, z \rangle = w \; \text{in } M \leadsto$ let $\langle y, z \rangle = w$ in let $x = N \; \text{in } M$

let $x = L$ in case $w$ of $[y]M$ or $[z]N \leadsto$ case $w$ of $[y] (\text{let } x = L \; \text{in } M)$ or $[z] (\text{let } x = L \; \text{in } N)$

let $x = N$ in let box $u = y \; \text{in } M \leadsto$ let box $u = y$ in let $x = N \; \text{in } M$

let $y = \lambda x : A. M$ in let $z = y \; N$ in $L \leadsto$ let $y = \lambda x : A. M$ in let $x = N$ in let $z = M \; \text{in } L$

let $x = \langle M, N \rangle$ in let $\langle y, z \rangle = x \; \text{in } L \leadsto$ let $y = M$ in let $z = N$ in let $x = \langle y, z \rangle \; \text{in } L$

let $x = \iota^A \lor^B (M)$ in case $x$ of $[y]N$ or $[z]L \leadsto$ let $y = M$ in let $x = \iota^A \lor^B (y) \; \text{in } N$

let $x = \iota^A \lor^B (M)$ in case $x$ of $[y]N$ or $[z]L \leadsto$ let $z = M$ in let $x = \iota^A \lor^B (z) \; \text{in } L$

let $x = \text{box } M$ in let box $u = x \; \text{in } N \leadsto$ let $u = M$ in let $x = \text{box } u \; \text{in } N$