# Some Weak Axiom Systems for CST

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.

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Part I: Rudimentary CST

Part II: Arithmetical CST

# Part I

Rudimentary CST

# The Axiom Systems CZF, BCST and RCST

- CZF is formulated in the first order language $\mathcal{L}_{\in}$ for intuitionistic logic with equality, having $\in$ as only non-logical symbol. It has the axioms of Extensionality, Emptyset, Pairing, Union and Infinity and the axiom schemes of $\Delta_0$-Separation, Strong Collection, Subset Collection and Set Induction. (CZF+ classical logic)$\equiv$ ZF.

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- BCST (Basic CST) is a weak subsystem of CZF. It uses Replacement instead of Strong Collection and otherwise only uses the axioms of Extensionality, Emptyset, Pairing, Union and Binary Intersection ($x \cap y$ is a set for sets $x, y$).

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- RCST (Rudimentary CST) is like BCST except that it uses the Replacement Rule (RR) instead of the Replacement Scheme.

- $\Delta_0$-Separation can be derived in RCST and so in BCST.

# The Replacement Rule

- Recall the Replacement Scheme:

$$\forall \underline{x} \forall x \{ (\forall z \in x) \exists ! y \phi[\underline{x}, z, y] \rightarrow \exists a \forall y (y \in a \ \leftrightarrow \ (\exists z \in x) \phi[\underline{x}, z, y]) \}$$

for each formula $\phi[\underline{x}, z, y]$, where $\underline{x}$ is a list $x_1, \ldots, x_n$ of distinct variables.

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Rudimentary CST (RCST):

Extensionality, Emptyset, Pairing, Union, Binary Intersection

and RR

# The Rudimentary Functions (à la Jensen)

Definition: *[Ronald Jensen (1972)] A function* $f : V^n \longrightarrow V$ *is* *Rudimentary if it is generated using the following schemata:*

**(a)** $f(\underline{x}) = x_i$

**(b)** $f(\underline{x}) = x_i - x_j$

**(c)** $f(\underline{x}) = \{x_i, x_j\}$

**(d)** $f(\underline{x}) = h(\underline{g}(\underline{x}))$

**(e)** $f(y, \underline{x}) = \cup_{z \in y} g(z, \underline{x})$

*where* $h : V^m \longrightarrow V$, $\underline{g} = g_1, \ldots, g_m : V^n \longrightarrow V$ *and* $g : V^{n+1} \longrightarrow V$ *are rudimentary and* $1 \leq i, j \leq n$.

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Note that $f(\underline{x}) = \emptyset = x_i - x_i$ is rudimentary; and so is $f(\underline{x}) = x_i \cap x_j = x_i - (x_i - x_j)$ using classical logic.

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Definition: *A function* $f : V^n \to V$ *is (CST)-Rudimentary if it is generated using the following schemata:*

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Proposition: *Using classical logic, the CST rudimentary functions coincide with Jensen's rudimentary functions.*

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Proposition: *If $R \subseteq V^{n+1}$ and $g : V^n \rightarrow V$ are rudimentary then so are $f : V^n \rightarrow V$ and $S \subseteq V^n$, where*

$$f(\underline{x}) = \{z \in g(\underline{x}) \mid R(z, \underline{x})\}$$

*and*

$$S(\underline{x}) \;\leftrightarrow\; R(g(\underline{x}), \underline{x}).$$

# The axiom system $RCST^*$, 1

- The language $\mathcal{L}^*_\in$ is obtained from $\mathcal{L}_\in$ by allowing individual terms $t$ generated using the following syntax equation:

$$t ::= z \mid \emptyset \mid \{t_1, t_2\} \mid t_1 \cap t_2 \mid \cup_{z \in t_1} t_2[z]$$

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Free occurences of $z$ in $t_2[z]$ become bound in $\cup_{z \in t_1} t_2[z]$. $RCST^*$ has the Extensionality axiom and the following comprehension axioms for the forms of term of $\mathcal{L}^*_\in$:

$$\begin{array}{lll} A1) & x \in \emptyset & \leftrightarrow \ \bot \\ A2) & x \in t_1 \cap t_2 & \leftrightarrow \ (x \in t_1 \wedge x \in t_2) \\ A3) & x \in \{t_1, t_2\} & \leftrightarrow \ (x = t_1 \vee x = t_2) \\ A4) & x \in \cup_{z \in t_1} t_2[z] & \leftrightarrow \ (\exists z \in t_1)\ (x \in t_2[z]) \end{array}$$

# The axiom system $RCST^*$, 2

Theorem: *For each term $t$ and each $\Delta_0$-formula $\phi[z]$ of $\mathcal{L}_\in^*$ there is a term $t'$ of $\mathcal{L}_\in^*$ such that $RCST^* \vdash (z \in t' \;\leftrightarrow\; z \in t \wedge \phi[z])$. We write $\{z \in t \mid \phi[z]\}$ for this term $t'$.*

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Some Definitions: Note: $(x, y) \in t \to x, y \in \cup\cup t$.

$$\{t\} \equiv \{t, t\}, \qquad (t_1, t_2) \equiv \{\{t_1\}, \{t_1, t_2\}\}$$

$$\cup t \equiv \cup_{z \in t} z, \qquad t_1 \cup t_2 \equiv \cup\{t_1, t_2\}$$

$$\{t_2[z] \mid z \in t_1\} \equiv \cup_{z \in t_1}\{t_2[z]\}$$

$$t_1 \times t_2 \equiv \cup_{x_1 \in t_1} \cup_{x_2 \in t_2} \{(t_1, t_2)\}$$

$$dom(t) \equiv \{x \in \cup\cup t \mid \exists y \in \cup\cup t \ (x, y) \in t\}$$

$$ran(t) \equiv \{y \in \cup\cup t \mid \exists x \in \cup\cup t \ (x, y) \in t\}$$

$$t_1{}'t_2 \equiv \cup\{y \in ran(t_1) \mid (t_2, y) \in t_1\}, \qquad t_1{}''t_2 \equiv \{t_1'x \mid x \in t_2\}$$

Note: $f'x = f(x)$ and $f''y = \{f(x) \mid x \in y\}$ if $f : a \to b$ and $x \in a, y \subseteq a$.

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Proposition: *We can associate with each term $t$ of $\mathcal{L}^*_\in$ a formula $\psi_t[y]$ of $\mathcal{L}_\in$ such that $RCST^* \vdash (y = t \;\leftrightarrow\; \psi_t[y])$ and $RCST \vdash \exists! y \psi_t[y]$.*

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Definition: $RCST_0$ *is the axiom system in the language $\mathcal{L}_\in$ with the Extensionality axiom and the axioms $\exists y \psi_t[y]$ for terms $t$ of $\mathcal{L}^*_\in$.*

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**Proposition:** *Every theorem of $RCST_0$ is a theorem of $RCST$ and $RCST^*$ is a conservative extension of $RCST_0$.*

# The axiom system $RCST^*$, 4

We simultaneously define formulae $\phi_t[x]$ such that $RCST^* \vdash (x \in t \leftrightarrow \phi_t[x])$ and $\psi_t[y]$ such that $RCST^* \vdash (y = t \leftrightarrow \psi_t[y])$ by structural recursion on terms $t$ of $\mathcal{L}^*_\in$:

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| $t$ | $\phi_t[x]$ |
|---|---|
| $z$ | $x \in z$ |
| $\emptyset$ | $\bot$ |
| $\{t_1, t_2\}$ | $\psi_{t_1}[x] \vee \psi_{t_2}[x]$ |
| $t_1 \cap t_2$ | $\phi_{t_1}[x] \wedge \phi_{t_2}[x]$ |
| $\cup_{z \in t_1} t_2[z]$ | $\exists z(\phi_{t_1}[z] \wedge \phi_{t_2[z]}[x])$ |

# The axiom system $RCST^*$, 5

If $\phi$ is a formula of $\mathcal{L}_\in^*$ let $\phi^\sharp$ be the formula of $\mathcal{L}_\in$ obtained from $\phi$ by replacing each atomic formula $t_1 = t_2$ by $\exists y(\psi_{t_1}[y] \wedge \psi_{t_2}[y])$ and each atomic formula $t_1 \in t_2$ by $\exists y(\psi_{t_1}[y] \wedge \phi_{t_2}[y])$.

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Proposition: *For each formula $\phi$ of $\mathcal{L}^*_\in$*

1. $RCST^* \vdash (\phi \leftrightarrow \phi^\sharp)$,
2. $\vdash (\phi \leftrightarrow \phi^\sharp)$ *if $\phi$ is a formula of $\mathcal{L}_\in$*,
3. $RCST^* \vdash \phi$ *implies* $RCST_0 \vdash \phi^\sharp$.

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Theorem: *[The Term Existence Property] If $RCST_0 \vdash \exists y \phi[y, \underline{x}]$ then $RCST^* \vdash \phi[t[\underline{x}], \underline{x}]$ for some term $t[\underline{x}]$ of $\mathcal{L}^*_\in$.*

Proof Idea: Use Friedman Realizability, as in Myhill (1973).

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Corollary: *The Replacement Rule is admissible for $RCST^*$ and hence $RCST \vdash \phi$ implies $RCST^* \vdash \phi$.*

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Corollary: *$RCST^*$ is a conservative extension of $RCST$.*
Proposition: *$RCST_0$ is finitely axiomatizable.*

The proof uses a constructive version of the result of Jensen that the rudimentary functions can be finitely generated using function composition.

# Part II

Arithmetical CST

# The class of natural numbers

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Definition: $Nat \equiv \{x \mid \forall y \in x^+(Trans(y) \wedge y \in \Gamma y)\}$ *where* $Trans(y) \equiv \forall z \in y\; z \subseteq y$.

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Note that $Nat$ is inductive.

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This axiom system has the same proof theoretic strength as Peano Arithmetic and is probably conservative over HA.

# Two Theorems of $ACST$

Theorem: *[The Finite AC Theorem] For classes $B, R$, if $A$ is a finite set such that $(\forall x \in A)(\exists y \in B)[(x, y) \in R]$ then there is a set function $f : A \to B$, such that $(\forall x \in A)[(x, f(x)) \in R]$.*

Proof: Use mathematical induction on the size of $A$.

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**Proof:** Use mathematical induction on the size of $A$.

**Theorem:** *[The Finitary Strong Collection Theorem] For classes $B, R$, if $A$ is a finitely enumerable set such that $(\forall x \in A)(\exists y \in B)[(x, y) \in R]$ there is a finitely enumerable set $B_0 \subseteq B$ such that*

$$(\forall x \in A)(\exists y \in B_0)[(x, y) \in R] \ \& \ (\forall y \in B_0)(\exists x \in A)[(x, y) \in R]$$

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**Proof:** Let $g : n \to A$ be a surjection, where $n \in Nat$, so that $(\forall k \in n)(\exists y \in B)[(g(k), y) \in R]$. By the finite AC theorem there is a function $f : n \to B$ such that, for all $m \in n$, $(g(m), f(m)) \in R$. The desired finitely enumerable set $B_0$ is $\{f(m) \mid m \in n\}$.

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- Any smallest $\Phi$-closed class is unique and is written $I(\Phi)$ and called the class inductively defined by $\Phi$

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Theorem: *[$ACST$] Each finitary inductive definition is generating.*

Example: The finitary inductive definition, having the steps $X/X$ for all finitely enumerable sets $X$, generates the class $HF$ of hereditarily finitely enumerable sets.

# The Primitive Recursion Theorem

- Theorem: Let $G_0 : B \to A$ and $F : Nat \times B \times A \to A$ be class functions, where $A, B$ are classes. Then there is a unique class function $G : Nat \times B \to A$ such that, for all $b \in B$ and $n \in Nat$,

$$(*) \quad \begin{cases} G(0, b) & = G_0(b), \\ G(n^+, b) & = F(n, b, G(n, b)), \end{cases}$$

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- Proof: : Let $G = I(\Phi)$, where $\Phi$ is the inductive definition with steps $\emptyset/((0, b), G_0(b))$, for $b \in B$, and $\{((n, b), x)\}/(n^+, F(n, b, x))$ for $(n, b, x) \in Nat \times B \times A$.

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- It is routine to show that $G$ is the unique required class function.
  ■

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  4. $Mult(n, m^+) = Plus(Mult(n, m), n)$.

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- Proof: Apply the Primitive Recursion theorem with $A = B = Nat$, first with $F(n, m, k) = k^+$ to obtain $Plus$ and then with $F(n, m, k) = Plus(k, n)$ to obtain $Mult$.
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- Proof: Apply the Primitive Recursion theorem with $A = B = Nat$, first with $F(n, m, k) = k^+$ to obtain $Plus$ and then with $F(n, m, k) = Plus(k, n)$ to obtain $Mult$.
  ■
- Using this result it is clear that there is an obvious standard interpretation of Heyting Arithmetic in $BCST_- + MathInd$.

# Some References

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