On the Bourbaki-Witt and Banach-Tarski Fixed-point Theorems

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A fixed-point theorem

**Theorem:** Let \((L, \leq)\) be a complete lattice and \(f : L \rightarrow L\) a **progressive map:** \(x \leq f(x)\) for all \(x \in L\). Then \(f\) has a fixed point.

**Proof.** Consider the least \(C \subseteq L\) closed under arbitrary suprema and \(f\). (\(C\) is the intersection of all subsets that are so closed.) Then \(y = \bigvee_{x \in C} f(x)\) is a fixed point because \(y \leq f(y) \leq \bigvee_{x \in C} f(x) = y\). QED.
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Classically, \(C\) is the chain obtained by iterations of \(f\), indexed by a sufficiently large ordinal,

\[
\bot \leq f(\bot) \leq f^2(\bot) \leq \cdots \leq f^\omega(\bot) \leq f^{\omega+1}(\bot) \leq \cdots
\]

so the theorem is classically valid for chain-complete posets (ccpo).

(A chain is a subset \(C\) such that \(x \leq y \lor y \leq x\) for all \(x, y \in C\).)
Fixed-point theorems for chain-complete posets

Theorem [Bourbaki-Witt, 1949 and 1951]:
A progressive map on a chain-complete poset has a fixed point.
Fixed-point theorems for chain-complete posets

Theorem [Bourbaki-Witt, 1949 and 1951]:
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Not to be confused with:

Theorem (Knaster-Tarski, 1955):
A monotone map on a chain-complete poset has a fixed point.
The intuitionistic status of BW and KT

Let \((P, \leq)\) be a poset and \(f : P \to P\) monotone or progressive. When does it have a fixed point, intuitionistically?

Does \(f\) have a fixed point?

- \(f\) progressive: yes [Tarski '55]
- \(P\) complete: yes
- \(P\) directed-complete: yes [Pataraia '97]
- \(P\) chain-complete: BW, KT

An observation by France Dacar: BW ⇔ BW dcpo and BW ⇒ KT.

We will show that (1), (2), and (3) may fail.
The intuitionistic status of BW and KT

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<tbody>
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<td>(P) complete</td>
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An observation by France Dacar: \(BW \Leftrightarrow BW^{dcpo}\) and \(BW \Rightarrow KT\). We will show that (1), (2), and (3) may fail.
Versions of Bourbaki-Witt

**Theorem:** The following are equivalent:

1. Progressive maps on ccpos have fixed points.
2. Ccpos have fixed-point operators for progressive maps.

Proof. $(2) \Rightarrow (1)$ is obvious. For $(1) \Rightarrow (2)$, let $(P, \leq)$ be a ccpo and let $\text{Prog}(P) = \{ f : P \to P | f \text{ progressive} \}$. The power $Q = \text{Prog}(P)$ with pointwise order is a ccpo. The map $F : Q \to Q$, defined by $F(h)(f) = f(h(f))$, is progressive and so has a fixed point $h$ by $(1)$. Then $h$ is a fixed-point operator on $\text{Prog}(P)$ since $h(f) = F(h)(f) = f(h(f))$. QED.
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\text{Prog}(P) = \{ f : P \to P \mid f \text{ progressive} \}.
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The power \(Q = P^{\text{Prog}(P)}\) with pointwise order is a ccpo. The map \(F : Q \to Q\), defined by \(F(h)(f) = f(h(f))\), is progressive and so has a fixed point \(h\) by (1). Then \(h\) is a fixed-point operator on \(\text{Prog}(P)\) since \(h(f) = F(h)(f) = f(h(f))\). QED.
Versions of anti-Bourbaki-Witt

**Theorem:** The following are equivalent:

1. There is a ccpo on which not all progressive maps have fixed points.
2. There is a ccpo and a progressive map on it without a fixed point.
Versions of anti-Bourbaki-Witt

**Theorem:** The following are equivalent:

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2. There is a ccpo and a progressive map on it without a fixed point.

**Proof.** Only (1) $\implies$ (2) requires proof.

Given $P$ as in (1), let $Q = P^{\text{Prog}(P)}$, ordered pointwise. Then $F : Q \to Q$ defined by $F(h)(f) = f(h(f))$ is progressive and without fixed point. QED.
Anti-Bourbaki-Witt and ordinals

The following are equivalent:
(1) There is a counter-example to Bourbaki-Witt theorem.
(2) Trichotomous ordinals form a set.
Anti-Bourbaki-Witt and ordinals

The following are equivalent:

1. There is a counter-example to Bourbaki-Witt theorem.
2. Trichotomous ordinals form a set.

Remark: $(\alpha, <)$ is a trichotomous ordinal when

1. $<$ is irreflexive and transitive,
2. $x < y \lor x = y \lor y < x$ for all $x, y \in \alpha$, and
3. $<$ is inductive: $(\forall y \in \alpha . (\forall x < y . \phi(x)) \Rightarrow \phi(y)) \Rightarrow \phi(z)$. 
The following are equivalent:

1. There is a counter-example to Bourbaki-Witt theorem.
2. Trichotomous ordinals form a set.

**Proof.** For (2) ⇒ (1) observe that if the class of ordinals \( \text{Ord} \) is a set then the successor map \( + : \text{Ord} \rightarrow \text{Ord} \) is progressive and has no fixed points. (NB: Successor will *not* be monotone because \( \text{Ord} \) is a dcpo!)

To prove (1) ⇒ (2), suppose \( P \) is a cppo and \( f : P \rightarrow P \) a progressive map without fixed points. We embed \( \text{Ord} \) into \( P \) via \( e : \text{Ord} \rightarrow P \) defined inductively by

\[
e(\alpha) = \bigvee_{\beta < \alpha} f(e(\beta)).
\]

The map \( e \) is injective because \( f \) has no fixed points. QED.
Failure of Bourbaki-Witt in realizability toposes

- In a realizability topos the ordinals form a set (an object).
  - For instance, in the effective topos the trichotomous ordinals are interpreted by the recursive ordinals and go up only to the Church-Kleene ordinal $\omega_{1}^{CK}$.
- Therefore, Bourbaki-Witt fails in realizability toposes.
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  - For instance, in the effective topos the trichotomous ordinals are interpreted by the recursive ordinals and go up only to the Church-Kleene ordinal \( \omega_{1}^{CK} \). 
- Therefore, Bourbaki-Witt fails in realizability toposes.
- This does not invalidate Knaster-Tarski, but a related counter example falsifies both Bourbaki-Witt and Knaster-Tarski at the same time.
  - For instance, in the effective topos \( \nabla \omega_{1} \) is chain-complete because all chains in the effective topos are countable. The successor map on \( \nabla \omega_{1} \) is both monotone and progressive, and has no fixed point. 
  - Here \( \omega_{1} \) is the least uncountable ordinal in \( \mathsf{Set} \) and \( \nabla \) is the inclusion of sets into the realizability topos as sheaves for the \( \neg \neg \)-coverage.
Theorem: If $(\phi^*, \phi_*) : \mathcal{E} \to \mathcal{F}$ is a geometric morphism between toposes and $\mathcal{F}$ validates Bourbaki-Witt then so does $\mathcal{E}$.
Transfer of Bourbaki-Witt along geometric morphisms

**Theorem:** If \((\phi^*, \phi_*) : \mathcal{E} \to \mathcal{F}\) is a geometric morphism between toposes and \(\mathcal{F}\) validates Bourbaki-Witt then so does \(\mathcal{E}\).

**Proof.** Suppose \(P\) is a ccpo in \(\mathcal{E}\) and \(f : P \to P\) progressive. It turns out that \(\phi_*P\) is a ccpo in \(\mathcal{F}\) and \(\phi_*h : \phi_*P \to \phi_*P\) progressive, therefore \(\mathcal{F}\) validates

\[\exists x \in \phi_*P . (\phi_*h)(x) = x.\]

The inverse image \(\phi^*\) preserves \(\exists\), hence \(\mathcal{E}\) validates

\[\exists y \in \phi^*(\phi_*P) . \phi^*(\phi_*h)(y) = y.\]

By naturality of the counit \(\epsilon_P : \phi^*(\phi_*P) \to P\) it follows that \(\epsilon_P(y)\) is a fixed point of \(h\):

\[h(\epsilon_P(y)) = \epsilon_P(\phi^*(\phi_*h)(y)) = \epsilon_P(y).\]

QED.
Bourbaki-Witt holds in sheaf toposes

Every sheaf topos $\mathcal{E}$, in fact every cocomplete topos, has a geometric morphism $(\phi^*, \phi_*) : \mathcal{E} \to \text{Set}$:

$$\phi^*(X) = \bigsqcup_{x \in X} 1 \quad \text{and} \quad \phi_*(A) = \text{Hom}_{\mathcal{E}}(1, A).$$
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- Bourbaki-Witt holds in sheaf toposes.
- Knaster-Tarski holds in sheaf toposes.
Bourbaki-Witt does not hold in the free topos

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- The free topos consists of definable objects and validates exactly the intuitionistically provable statements.
- Bourbaki-Witt cannot have a definable counter-example as that would give one in Set.
- Perhaps Bourbaki-Witt is valid in the free topos, i.e., for each definable ccppo we can prove that every definable progresive map has a fixed point?
Bourbaki-Witt does not hold in the free topos

**Theorem:** There is a cpo in the free topos for which the Bourbaki-Witt theorem is not provable.
Bourbaki-Witt does not hold in the free topos

**Theorem:** There is a ccpo in the free topos for which the Bourbaki-Witt theorem is not provable.

*Proof.* The idea is to define a ccpo whose interpretation in the effective topos is $\text{Ord}$. Let $P$ consist of ordinals that are $\neg\neg$-stable quotients of $\neg\neg$-stable subsets of $\mathbb{N}$. This is definable — we just did it — but we cannot prove that it is ccpo, because in $\text{Set}$ it is interpreted by $\omega_1$, which is not chain-complete.

Let $S = \{ x \in 1 \mid P \text{ is a ccpo} \}$ and $Q = P^S$. Then $Q$ is a ccpo. Its interpretation in the effective topos is $\text{Ord}$ (and in $\text{Set}$ it is $P^0 = 1$).

We cannot prove that (the exponent by $S$ of) the successor map $+: Q \to Q$ has a fixed point, otherwise it would have one in the effective topos. QED.
Bourbaki-Witt does not imply existence of ordinals

- A counterexample to BW implies “few ordinals”.
- Perhaps validity of BW implies “many ordinals”?
  - By “many” we mean sufficiently many to reach fixed points of progressive maps by iteration.
Bourbaki-Witt does not imply existence of ordinals

**Theorem:** There is a topos which validates BW and contains a progressive $f : P \to P$ on a ccpo $P$ for which no amount of iteration along ordinals reaches a fixed point.
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**Theorem:** There is a topos which validates BW and contains a progresive $f : P \to P$ on a ccpo $P$ for which no amount of iteration along ordinals reaches a fixed point.

**Proof.** The topos $\mathcal{E}$ in question is the free topos satisfying BW with a generic ccpo $P$ and a generic progressive $f : P \to P$. No ordinal $\alpha$ in $\mathcal{E}$ suffices to reach the fixed point of $f$. If interested, ask for details.

$$\begin{align*}
\mathcal{E} & \to \text{Set}^{0 \to 1} \\
\alpha, P, f & \to (\lambda_0, \lambda_1), (\{\star\}, L), (\text{id}_{\{\star\}}, s) \\
\lambda_0, \{\star\}, \text{id}_{\{\star\}} & \downarrow
\end{align*}$$

where $L = \lambda_0 + 1$, $s : L \to L$, $s(x) = \min(x + 1, \lambda_0)$. Observe that $\lambda_1 \leq \lambda_0$, hence $(\lambda_0, \lambda_1)$-many iterations are not enough for $s$ to reach a fixed point.
Two questions

1. *Does Knaster-Tarski imply Bourbaki-Witt?*
   - Yes? Give a proof. NB: the proof must work for ccpo’s but fail for dcpo’s.
   - No? Give a model which validates Knaster-Tarski and falsifies Bourbaki-Witt.

2. *Is there a constructive version of the Bourbaki-Witt theorem?*
   - Weaken “chain” or strengthen “progressive”, but how?