On the Bourbaki-Witt and Banach-Tarski Fixed-point Theorems

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A fixed-point theorem

Theorem: *Let* $(L, \leq)$ *be a complete lattice and* $f : L \to L$ *a* ***progressive map****:* $x \leq f(x)$ *for all* $x \in L$*. Then* f *has a fixed point.*

Proof. Consider the least $C \subseteq L$ closed under arbitrary suprema and f. (C is the intersection of all subsets that are so closed.) Then $y = \bigvee_{x \in C} f(x)$ is a fixed point because $y \leq f(y) \leq \bigvee_{x \in C} f(x) = y$. QED.

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Classically, C is the chain obtained by iterations of f, indexed by a sufficiently large ordinal,

$$\bot \leq f(\bot) \leq f^2(\bot) \leq \cdots \leq f^\omega(\bot) \leq f^{\omega+1}(\bot) \leq \cdots$$

so the theorem is classically valid for *chain-complete posets (ccpo)*.

(A *chain* is a subset C such that $x \leq y \vee y \leq x$ for all $x, y \in C$.)

Fixed-point theorems for chain-complete posets

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Not to be confused with:

Theorem (Knaster-Tarski, 1955):
A ***monotone*** *map on a chain-complete poset has a fixed point.*

The intuitionistic status of BW and KT

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Does f have a fixed point?		
	f progressive	f monotone
P complete	yes	yes [Tarski '55]
P directed-complete	$\mathrm{BW}^{\mathrm{dcpo}}$	yes [Pataraia '97]
P chain-complete	BW	KT

An observation by France Dacar: $\mathrm{BW} \Leftrightarrow \mathrm{BW}^{\mathrm{dcpo}}$ and $\mathrm{BW} \Rightarrow \mathrm{KT}$.

We will show that (1), (2), and (3) may fail.

Versions of Bourbaki-Witt

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Proof. (2) $\Rightarrow$ (1) is obvious. For (1) $\Rightarrow$ (2), let $(P, \leq)$ be a ccpo and let

$$\mathrm{Prog}(P) = \{f : P \to P \mid f \text{ progressive}\}.$$

The power $Q = P^{\mathrm{Prog}(P)}$ with pointwise order is a ccpo. The map $F : Q \to Q$, defined by $F(h)(f) = f(h(f))$, is progressive and so has a fixed point h by (1). Then h is a fixed-point operator on $\mathrm{Prog}(P)$ since $h(f) = F(h)(f) = f(h(f))$. QED.

Versions of anti-Bourbaki-Witt

Theorem: *The following are equivalent:*

(1) *There is a ccpo on which not all progressive maps have fixed points.*

(2) *There is a ccpo and a progresive map on it without a fixed point.*

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Theorem: *The following are equivalent:*

(1) *There is a ccpo on which not all progressive maps have fixed points.*

(2) *There is a ccpo and a progresive map on it without a fixed point.*

Proof. Only (1) $\Rightarrow$ (2) requires proof.

Given P as in (1), let $Q = P^{\mathrm{Prog}(P)}$, ordered pointwise. Then $F : Q \to Q$ defined by $F(h)(f) = f(h(f))$ is progressive and without fixed point. QED.

Anti-Bourbaki-Witt and ordinals

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(1) There is a counter-example to Bourbaki-Witt theorem.

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Remark: $(\alpha, <)$ is a *trichotomous ordinal* when

1. $<$ is irreflexive and transitive,
2. $x < y \lor x = y \lor y < x$ for all $x, y \in \alpha$, and
3. $<$ is inductive: $(\forall\, y \in \alpha\,.\,(\forall\, x < y\,.\,\phi(x)) \Rightarrow \phi(y)) \Rightarrow \phi(z)$.

Anti-Bourbaki-Witt and ordinals

The following are equivalent:

(1) There is a counter-example to Bourbaki-Witt theorem.

(2) Trichotomous ordinals form a set.

Proof. For (2) $\Rightarrow$ (1) observe that if the class of ordinals Ord is a set then the successor map $^{+} : \mathsf{Ord} \to \mathsf{Ord}$ is progressive and has no fixed points. (NB: Successor will *not* be monotone because Ord is a dcpo!)

To prove (1) $\Rightarrow$ (2), suppose P is a cppo and $f : P \to P$ a progressive map without fixed points. We embed Ord into P via $e : \mathsf{Ord} \to P$ defined inductively by

$$e(\alpha) = \bigvee\nolimits_{\beta<\alpha} f(e(\beta)).$$

The map e is injective because f has no fixed points. QED.

Failure of Bourbaki-Witt in realizability toposes

- In a realizability topos the ordinals form a set (an object).
  - For instance, in the effective topos the trichotomous ordinals are interpreted by the recursive ordinals and go up only to the Church-Kleene ordinal ω_1^{CK}.
- Therefore, Bourbaki-Witt fails in realizability toposes.

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- In a realizability topos the ordinals form a set (an object).
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- Therefore, Bourbaki-Witt fails in realizability toposes.
- This does not invalidate Knaster-Tarski, but a related counter example falsifies both Bourbaki-Witt and Knaster-Tarski at the same time.
  - For instance, in the effective topos $\nabla\omega_1$ is chain-complete because all chains in the effective topos are countable. The successor map on $\nabla\omega_1$ is both monotone and progressive, and has no fixed point.
  - Here ω_1 is the least uncountable ordinal in Set and ∇ is the inclusion of sets into the realizability topos as sheaves for the $\neg\neg$-coverage.

Transfer of Bourbaki-Witt along geometric morphisms

Theorem: If $(\phi^*, \phi_*) : \mathcal{E} \to \mathcal{F}$ is a geometric morphism between toposes and $\mathcal{F}$ validates Bourbaki-Witt then so does $\mathcal{E}$.

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Proof. Suppose P is a ccpo in $\mathcal{E}$ and $f : P \to P$ progressive. It turns out that $\phi_* P$ is a ccpo in $\mathcal{F}$ and $\phi_* h : \phi_* P \to \phi_* P$ progressive, therefore $\mathcal{F}$ validates

$$\exists\, x \in \phi_* P \,.\, (\phi_* h)(x) = x.$$

The inverse image ϕ^* preserves $\exists$, hence $\mathcal{E}$ validates

$$\exists\, y \in \phi^*(\phi_* P) \,.\, \phi^*(\phi_* h)(y) = y.$$

By naturality of the counit $\epsilon_P : \phi^*(\phi_* P) \to P$ it follows that $\epsilon_P(y)$ is a fixed point of h:

$$h(\epsilon_P(y)) = \epsilon_P(\phi^*(\phi_* h)(y)) = \epsilon_P(y).$$

QED.

Bourbaki-Witt holds in sheaf toposes

- Every sheaf topos $\mathcal{E}$, in fact every cocomplete topos, has a geometric morphism $(\phi^*, \phi_*) : \mathcal{E} \to \mathsf{Set}$:

$$\phi^*(X) = \coprod_{x \in X} 1 \qquad \text{and} \qquad \phi_*(A) = \mathsf{Hom}_{\mathcal{E}}(1, A).$$

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- Bourbaki-Witt holds in sheaf toposes.
- Knaster-Tarski holds in sheaf toposes.

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- The free topos consists of definable objects and validates exactly the intuitionistically provable statements.
- Bourbaki-Witt cannot have a definable counter-example as that would give one in Set.
- Perhaps Bourbaki-Witt is valid in the free topos, i.e., for each definable ccpo we can prove that every definable progresive map has a fixed point?

Bourbaki-Witt does not hold in the free topos

Theorem: *There is a ccpo in the free topos for which the Bourbaki-Witt theorem is not provable.*

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Proof. The idea is to define a ccpo whose interpretation in the effective topos is Ord.

Take P to consist of ordinals that are $\neg\neg$-stable quotients of $\neg\neg$-stable subsets of $\mathbb{N}$. This is definable — we just did it — but we cannot prove that it is ccpo, because in Set it is interpreted by ω_1, which is not chain-complete.

Let $S = \{x \in 1 \mid P \text{ is a ccpo}\}$ and $Q = P^S$. Then Q is a ccpo. Its interpretation in the effective topos is Ord (and in Set it is $P^\emptyset = 1$).

We cannot prove that (the exponent by S of) the successor map $^+ : Q \to Q$ has a fixed point, otherwise it would have one in the effective topos. QED.

Bourbaki-Witt does not imply existence of ordinals

- A counterexample to BW implies “few ordinals”.
- Perhaps validity of BW implies “many ordinals”?
  - By “many” we mean sufficiently many to reach fixed points of progressive maps by iteration.

Bourbaki-Witt does not imply existence of ordinals

Theorem: *There is a topos which validates BW and contains a progresive $f : P \to P$ on a ccpo P for which no amount of iteration along ordinals reaches a fixed point.*

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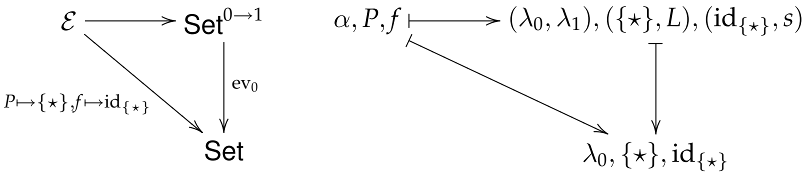
Proof. The topos $\mathcal{E}$ in question is the free topos satisfying BW with a generic ccpo P and a generic progressive $f : P \to P$. No ordinal α in $\mathcal{E}$ suffices to reach the fixed point of f. If interested, ask for details.



where $L = \lambda_0 + 1$, $s : L \to L$, $s(x) = \min(x + 1, \lambda_0)$. Observe that $\lambda_1 \leq \lambda_0$, hence (λ_0, λ_1)-many iterations are not enough for s to reach a fixed point.

Two questions

1. *Does Knaster-Tarski imply Bourbaki-Witt?*
   - Yes? Give a proof. NB: the proof must work for ccpo's but fail for dcpo's.
   - No? Give a model which validates Knaster-Tarski and falsifies Bourbaki-Witt.
2. *Is there a constructive version of the Bourbaki-Witt theorem?*
   - Weaken "chain" or strengthen "progressive", but how?