Aligning the weak König lemma, the uniform continuity theorem, and Brouwer's fan theorem

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Definitions

$\{0,1\}^*$ the set of all finite binary sequences u, v, w

$|u|$ the length of u

$u * v$ the concatenation of u and v

$\{0,1\}^{\mathbb{N}}$ *Cantor space*, the set of all infinite binary sequences α, β, γ

$\overline{\alpha}n = (\alpha(0), \ldots, \alpha(n-1))$

WKL

A subset T of $\{0,1\}^*$ is

- *detachable* if $\forall u (u \in T \vee u \notin T)$;
- *closed under restriction* if

$$\forall u, v\, (u * v \in T \rightarrow u \in T)\, ;$$

- an *infinite binary tree* if
  - it is detachable
  - it is closed under restriction
  - we have

$$\forall n\, \exists u\, (|u| = n \wedge u \in T)\, .$$

WKL

Let $\mathcal{T}$ denote the set of all infinite binary trees. Fix $T \in \mathcal{T}$. We define

$$\mathcal{A}_T = \{\alpha \mid \forall n\, (\overline{\alpha}n \in \mathrm{T})\}\,.$$

A sequence $\alpha \in \mathcal{A}_T$ is called an *infinite path* of T.

The *weak König lemma* is the following axiom.

WKL Every infinite tree has an infinite path.

FAN

A detachable subset B of $\{0,1\}^*$ is

- a *bar* if $\forall\alpha\,\exists n\,(\overline{\alpha}n \in B)$;
- a *uniform bar* if $\exists N\,\forall\alpha\,\exists n \leq N\,(\overline{\alpha}n \in B)$.

Brouwer's fan theorem for detachable bars reads as follows.

FAN Every bar is a uniform bar.

Topology on $\{0,1\}^{\mathbb{N}}$

On $\{0,1\}^{\mathbb{N}}$ we have the metric

$$d(\alpha, \beta) = \inf \left\{ 2^{-n} \mid \overline{\alpha} n = \overline{\beta} n \right\} .$$

A function $F : \{0,1\}^{\mathbb{N}} \to \mathbb{N}$ is *pointwise continuous* if

$$\forall \alpha \, \exists n \, \forall \beta \left(\overline{\alpha} n = \overline{\beta} n \to F(\alpha) = F(\beta) \right)$$

and *uniformly continuous* if

$$\exists N \, \forall \alpha, \beta \left(\overline{\alpha} N = \overline{\beta} N \to F(\alpha) = F(\beta) \right) .$$

UC

The *uniform continuity theorem* is the following axiom.

UC Every pointwise continuous function $F : \{0,1\}^{\mathbb{N}} \to \mathbb{N}$ is uniformly continuous.

We say that $\mathcal{A}_T$ is *decidable* if

$$\forall \alpha \, (\alpha \in \mathcal{A}_T \vee \exists n \, (\overline{\alpha} n \notin T)) \, .$$

For u of length n we define

$$O_u = \{\alpha \mid \overline{\alpha} n = u\} \, .$$

Let us define the following axioms.

($\sharp$) If $T \in \mathcal{T}$ then $\mathcal{A}_T$ is inhabited.

($\sharp\sharp$) Fix $T \in \mathcal{T}$ such that $\mathcal{A}_T$ is decidable and open. Then $\mathcal{A}_T$ is inhabited. Moreover, we have

$$\forall u \, (u \in T) \vee \exists u \, (u \notin T) \, .$$

($\sharp\sharp\sharp$) Fix $T \in \mathcal{T}$ and suppose that $\mathcal{A}_T$ is decidable. Assume further that

$$\forall \alpha \in \mathcal{A}_T \, \exists n \, (\mathcal{A}_T = O_{\overline{\alpha} n}) \, .$$

Then $\mathcal{A}_T$ is inhabited.

Proposition

We have

$$(\sharp) \;\Rightarrow\; (\sharp\sharp) \;\Rightarrow\; (\sharp\sharp\sharp)\,.$$

Proposition

1. $(\sharp)$ *is equivalent to* WKL
2. $(\sharp\sharp)$ *is equivalent to* UC
3. $(\sharp\sharp\sharp)$ *is equivalent to* FAN