On the Power of Choice

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Outline

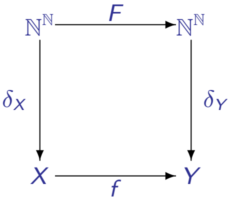
Realizer

Definition

A multi-valued function $f :\subseteq X \rightrightarrows Y$ on represented spaces (X, δ_X) and (Y, δ_Y) is realized by a function $F :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ if

$$\delta_Y F(p) \in f \delta_X(p)$$

for all $p \in \mathrm{dom}(f\delta_X)$. We write $F \vdash f$ in this situation.



Weihrauch Reducibility

Definition

For two multi-valued functions f and g on represented spaces we say that f is Weihrauch reducible to g, in symbols $f \leq_{\mathrm{W}} g$, if there are computable functions H and K such that

$$G \vdash g \implies H\langle \mathrm{id}, GK \rangle \vdash f$$

holds for all G.

That means that there is a uniform way to transform each realizer G of g into a realizer F of f in the given way.

Proposition

Weihrauch reducibility is a preorder on the set of multi-valued functions (on some given category of represented spaces) and it induces a partial order.

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Algebraic Operations in the Weihrauch Lattice

Definition

Let $f :\subseteq X \rightrightarrows Y$ and $g :\subseteq W \rightrightarrows Z$ be multi-valued maps. Then we consider the natural operations

- $f \times g :\subseteq X \times W \rightrightarrows Y \times Z$ (product)
- $f \sqcup g :\subseteq X \sqcup W \rightrightarrows Y \sqcup Z$ (coproduct)
- $f \oplus g :\subseteq X \times W \rightrightarrows Y \sqcup Z$ (sum)
- $f^* :\subseteq X^* \rightrightarrows Y^*, f^* = \bigsqcup_{i=0}^{\infty} f^i$ (star)
- $\widehat{f} :\subseteq X^{\mathbb{N}} \rightrightarrows Y^{\mathbb{N}}, \widehat{f} = \bigtimes_{i=0}^{\infty} f$ (parallelization)

Proposition

Weihrauch reducibility induces a (bounded) lattice with the sum $\oplus$ as infimum and the coproduct $\sqcup$ as supremum and parallelization and the star operation are closure operators in this lattice.

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The Choice Operation

Definition

We define the choice operation

$$\mathsf{C}_X :\subseteq \mathcal{A}_-(X) \rightrightarrows X, A \mapsto A$$

for every represented space X. Here

$$\mathcal{A}_-(X) := \{A \subseteq X : A \text{ closed}\}$$

is the hyperspace of closed subsets with respect to negative information (the upper Fell topology = dual of the Scott topology).

That is, choice C_X is an operation that takes as input a description of what does *not* constitute a solution and has to find a solution.
By UC_X we denote unique choice, i.e. the restriction of C_X to singletons.

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Finite Choice

Definition

For each natural number $n \in \mathbb{N}$ we write for short

$$\mathbf{n} := \mathsf{C}_{\{0,\dots,n-1\}}.$$

That is $\mathbf{n}$ reflects choice between n alternatives.

Proposition

- $\mathbf{0} = \mathsf{C}_\emptyset$ *is a neutral element with respect to the coproduct* $\sqcup$ *and acts like a zero with respect to products* $\times$
- $\mathbf{0} \leq_{\mathrm{W}} f$ *for all* f*, i.e.* $\mathbf{0}$ *is the bottom element*
- $\mathbf{1} = \mathsf{C}_{\{0\}} \equiv_{\mathrm{W}} \mathbf{0}^*$ *is a neutral element with respect to the product* $\times$

The Weihrauch lattice together with $\sqcup, \times, ^*, \mathbf{0}, \mathbf{1}$ forms a commutative semiring and a continuous Kleene algebra.

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Characterization of Computability

Theorem

For all f the following statements are equivalent:

- $f \leq_{\mathrm{W}} \mathbf{1}$
- f *is computable.*

$$\mathbf{1} = \mathrm{C}_{\{0\}} \equiv_{\mathrm{W}} \mathbf{0}^*$$

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$$\mathbf{0} = \mathrm{C}_{\emptyset}$$

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The entire Turing semi-lattice can be embedded inbetween $\mathbf{0}$ and $\mathbf{1}$ (with order reversed).

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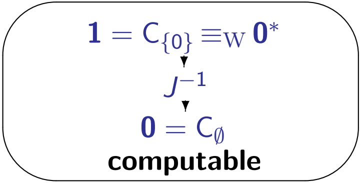

Theorem

The entire Turing semi-lattice can be embedded inbetween $\mathbf{0}$ *and* $\mathbf{1}$ *(with order reversed).*

Idempotency and Pointedness

Definition

- f is called pointed if $\mathbf{1} \leq_{\mathrm{W}} f$,
- f is called idempotent if $f \equiv_{\mathrm{W}} f \times f$.

Proposition

For pointed f, g are pointed and $f \sqcup g$ is idempotent, then

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Proposition

A pointed f is idempotent if and only if $f^* \equiv_{\mathrm{W}} f$.

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Binary Choice and LLPO

Example

- Binary choice $\mathbf{2} = \mathsf{C}_{\{0,1\}}$ could receive as a potential input:

$$\bot, \bot, \bot, 1, 1, \bot, 1, 1, 1, \ldots$$

- Here $\bot$ stands for "no information". As soon as the information 1 appears, it is clear that the only possible remaining choice is 0.
- This is similar to the "lesser limited principle of omniscience" LLPO.

Proposition

$$\mathsf{LLPO} \equiv_{\mathrm{W}} \mathbf{2}.$$

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Coproducts, Products and Compositional Products

Definition

For f and g we define the compositional product $f * g$ by

$$f * g = \sup\{f_0 \circ g_0 : f_0 \leq_{\mathrm{W}} f \text{ and } g_0 \leq_{\mathrm{W}} g\}.$$

Proposition

For pointed f, g we obtain

$$f \oplus g \leq_{\mathrm{W}} f \sqcup g \leq_{\mathrm{W}} f \times g \leq_{\mathrm{W}} f * g.$$

Proof.

Here the last reduction follows from
$f \times g = (f \times \mathrm{id}) \circ (\mathrm{id} \times g) \leq_{\mathrm{W}} f * g$. □

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Products of Choice and Weihrauch Arithmetic

Proposition

For non-empty A, B we obtain

$$\mathsf{C}_A \sqcup \mathsf{C}_B \leq_{\mathrm{W}} \mathsf{C}_A \times \mathsf{C}_B \leq_{\mathrm{W}} \mathsf{C}_{A \times B}.$$

Corollary

$$\mathbf{n} \times \mathbf{k} \leq_{\mathrm{W}} \mathbf{n} \cdot \mathbf{k}$$

for all $n, k \in \mathbb{N}$.

Proposition

$$\mathbf{2} \times \mathbf{2} \not\equiv_{\mathrm{W}} \mathbf{4}.$$

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Natural Choice and Finitely Many Mind Changes

Example

- Natural number choice $\mathsf{C}_{\mathbb{N}}$ could receive as a potential input:

$$5, 112, 3, 5, 23, 0, 42, 1, 25, \ldots$$

- This is a discontinuous operation, however, it can be computed with finitely many mind changes.

Theorem

For all f the following statements are equivalent:

- $f \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$
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The Baire Category Theorem

Definition

Let X be a non-empty computable metric space. We define

$$\mathsf{BCT} :\subseteq \mathcal{A}_-(X)^{\mathbb{N}} \rightrightarrows \mathbb{N}, (A_i)_{i\in\mathbb{N}} \mapsto \{n \in \mathbb{N} : A_n^\circ \neq \emptyset\}$$

with $\mathrm{dom}(\mathsf{BCT}) = \{(A_i)_{i\in\mathbb{N}} : X = \bigcup_{i=0}^{\infty} A_i\}$.

Theorem

$$\mathsf{BCT} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{UC}_{\mathbb{N}}.$$

Other equivalent theorems:

- Banach's Inverse Mapping Theorem,
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Surjections and Idempotency

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Let A and B be represented spaces and let $s : A \to B$ be a computable surjection. Then $\mathsf{C}_B \leq_{\mathrm{W}} \mathsf{C}_A$.

Corollary

Let A be a represented space. If there is a computable surjection $s : A \to A^2$, then C_A is idempotent, i.e. $\mathsf{C}_A \times \mathsf{C}_A \equiv_{\mathrm{W}} \mathsf{C}_{A \times A} \equiv_{\mathrm{W}} \mathsf{C}_A$.

Corollary

The choice principles $\mathsf{C}_{\mathbb{N}}$, $\mathsf{C}_{\{0,1\}^{\mathbb{N}}}$, $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ and $\mathsf{C}_{\mathbb{N} \times \{0,1\}^{\mathbb{N}}}$ are idempotent.

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Let A and B be represented spaces and let $s : A \to B$ be a computable surjection. Then $\mathsf{C}_B \leq_{\mathrm{W}} \mathsf{C}_A$.

Corollary

Let A be a represented space. If there is a computable surjection $s : A \to A^2$, then C_A is idempotent, i.e. $\mathsf{C}_A \times \mathsf{C}_A \equiv_{\mathrm{W}} \mathsf{C}_{A \times A} \equiv_{\mathrm{W}} \mathsf{C}_A$.

Corollary

The choice principles $\mathsf{C}_{\mathbb{N}}$, $\mathsf{C}_{\{0,1\}^{\mathbb{N}}}$, $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$ and $\mathsf{C}_{\mathbb{N} \times \{0,1\}^{\mathbb{N}}}$ are idempotent.

Non-Deterministic Computability

Theorem

Let X and Y be represented spaces, $A \subseteq \mathbb{N}^{\mathbb{N}}$ and let $f :\subseteq X \rightrightarrows Y$ be a multi-valued function. Then the following are equivalent:

- $f \leq_{\mathrm{W}} \mathsf{C}_A$,
- *f is non-deterministically computable with advice space A.*

Definition

A function $f :\subseteq X \rightrightarrows Y$ is said to be non-deterministically computable with advice space $A \subseteq \mathbb{N}^{\mathbb{N}}$,

- if there is a suitable advice $r \in A$ for each input that leads to a correct result,
- if unsuitable advices $r \in A$ for each input can be recognized in finite time.

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Independent Choice

Theorem

Let $A, B \subseteq \mathbb{N}^{\mathbb{N}}$ *be non-empty. Then* $\mathsf{C}_A * \mathsf{C}_B \leq_{\mathrm{W}} \mathsf{C}_{A\times B}$.

Corollary

Let $A \subseteq \mathbb{N}^{\mathbb{N}}$ *be a subspace of Baire space. If there is a computable surjection* $s : A \to A^2$, *then* C_A *is closed under composition and idemotent, i.e.* $\mathsf{C}_A \times \mathsf{C}_A \equiv_{\mathrm{W}} \mathsf{C}_A * \mathsf{C}_A \equiv_{\mathrm{W}} \mathsf{C}_{A\times A} \equiv_{\mathrm{W}} \mathsf{C}_A$.

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Choice on Computable Metric Spaces

Corollary

Let X be a computable Polish space. Then $\mathsf{C}_X \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$. If, additionally, X is computably compact, then $\mathsf{C}_X \leq_{\mathrm{W}} \mathsf{C}_{\{0,1\}^{\mathbb{N}}}$.

Proposition

Let A and B be computable metric spaces and let $\iota : A \to B$ be a computable embedding such that $\mathrm{range}(\iota)$ is co-c.e. closed in B. Then $\mathsf{C}_A \leq_{\mathrm{W}} \mathsf{C}_B$.

Corollary

Let X be a computably compact metric space, which is non-empty and has no isolated points, then $\mathsf{C}_{\{0,1\}^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{C}_X$.

Corollary

$\mathsf{C}_{\{0,1\}^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{C}_{[0,1]} \equiv_{\mathrm{W}} \mathsf{C}_{[0,1]^{\mathbb{N}}}$.

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Weak Kőnig's Lemma

Example

- Cantor choice $C_{\{0,1\}^{\mathbb{N}}}$ could receive as a potential input a sequence of finite words:

$$0111000, 01000, 010100001111000, \ldots$$

- The goal is to find an infinite word that does not have any of these words as prefix.

Theorem

$$\mathsf{WKL} \equiv_W C_{\{0,1\}^{\mathbb{N}}} \equiv_W \widehat{C_{\{0,1\}}} = \widehat{\mathbf{2}}.$$

Another equivalent theorem is:

- The Hahn-Banach Theorem (Gherardi & Marcone)

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Compact Choice

Theorem

For all f the following statements are equivalent:

- $f \leq_{\mathrm{W}} \mathsf{C}_{\{0,1\}^{\mathbb{N}}}$
- f *is weakly computable.*

Theorem

Any single-valued function $f : X \to Y$ on computable metric space that is weakly computable is already computable.

Corollary

$$\mathsf{UC}_{\{0,1\}^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{C}_{\{0\}} \equiv_{\mathrm{W}} \mathbf{1}.$$

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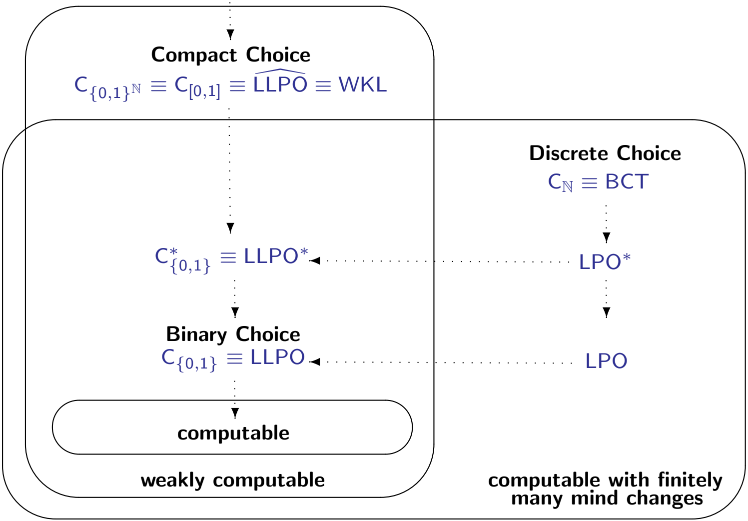
Weak Computability and Finitely Many Mind Changes



Locally Compact Choice

Proposition

Let X be a computable K_σ–space. Then
$\mathsf{C}_X \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} \times \mathsf{C}_{\{0,1\}^{\mathbb{N}}} \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}\times\{0,1\}^{\mathbb{N}}}$.

Corollary

$\mathsf{C}_{\mathbb{R}^k} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{R}} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}\times\{0,1\}^{\mathbb{N}}} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}} \times \mathsf{C}_{\{0,1\}^{\mathbb{N}}}$ *for all* $k \geq 1$.

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If $f : X \to Y$ is a single-valued function on computable metric spaces and $f \leq_{\mathrm{W}} \mathsf{C}_{\{0,1\}^{\mathbb{N}}} \times \mathsf{C}_{\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{R}}$, then $f \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{N}}$.

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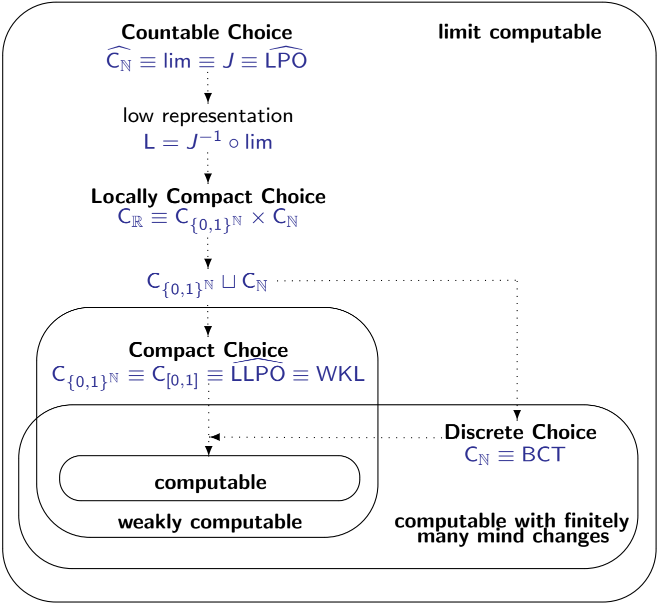
Choice and Limit Computability



The Uniform Low Basis Theorem

Theorem

$\mathsf{C}_{\{0,1\}^{\mathbb{N}}}$ and $\mathsf{C}_{\mathbb{R}}$ are low computable.

Corollary (Low Basis Theorem of Jockusch and Soare)

Each co-c.e. closed subset $A \subseteq \{0,1\}^{\mathbb{N}}$ has a low point $p \in A$, i.e. a point such that $p' \leq_{\mathrm{T}} \emptyset'$.

Theorem

For all f the following statements are equivalent:

- $f \leq_{\mathrm{sW}} \mathsf{L} = J^{-1} \circ \lim$
- *f is low computable.*

Corollary

The Brouwer Fixed Point Theorem and the Hahn-Banach Theorem are low computable.

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Effective Borel Measurability

Corollary

If X is a Polish space, then there is an oracle such that

$$\text{either } \mathsf{C}_X \leq_{\mathrm{W}} \mathsf{C}_{\mathbb{R}} \text{ or } \mathsf{C}_X \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^{\mathbb{N}}}$$

relatively to that oracle (i.e. with continuous reductions).

Theorem

Let X and Y be computable Polish spaces and let $f : X \to Y$ be a function. Then the following are equivalent:

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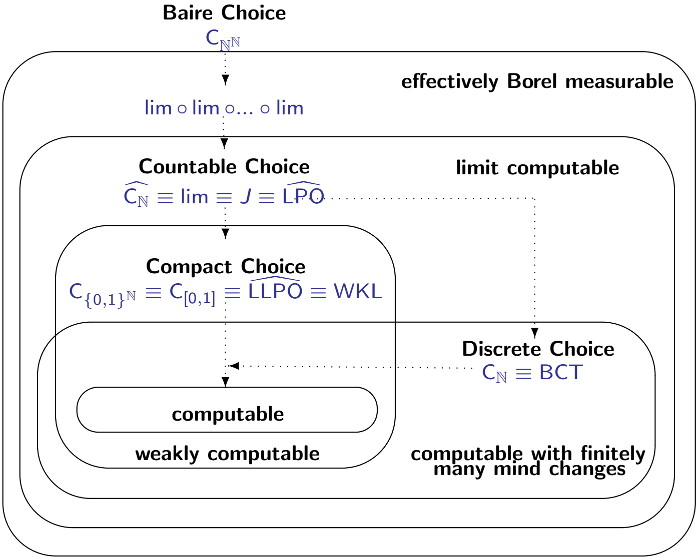
Survey on Choice Classes



Open Problems

- Another conjecture (with Arno Pauly and Matthew de Brecht) is that $\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$, but we have no proof yet.
- Is the Weihrauch lattice a Brouwerian algebra (Heyting lattice)? The answer is “yes” for total Weihrauch reducibility but not known for the ordinary reducibility.
- In a current joint project with Arno Pauly and Stephane Le Roux we are trying to classify the Brouwer Fixed Point Theorem BFT more precisely.
- It is known that $\mathsf{C}_{[0,1]} \equiv_{\mathrm{W}} \mathsf{IVT} \equiv_{\mathrm{W}} \mathsf{BFT}_1 <_{\mathrm{W}} \mathsf{WKL}$, i.e. the one-dimensional Brouwer Fixed Point Theorem is equivalent to the Intermediate Value Theorem and strictly below Weak König's Lemma.
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- In this context, one would wish to classify connected closed choice.

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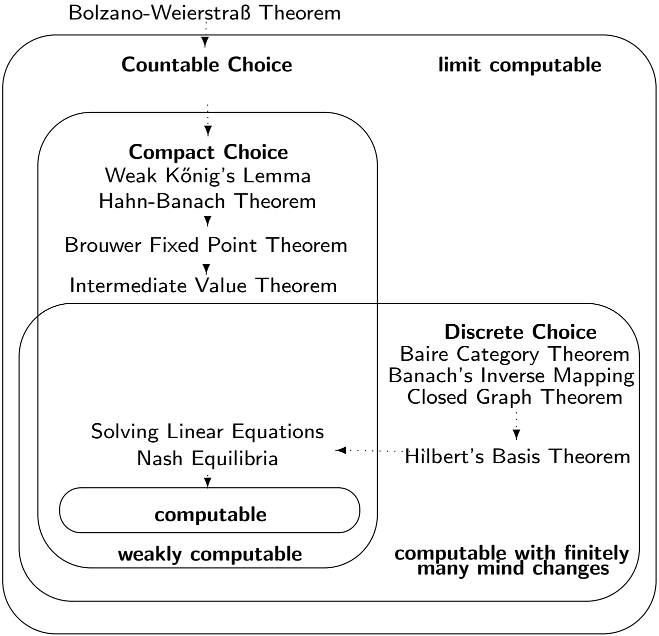
Open Problems

- Another conjecture (with Arno Pauly and Matthew de Brecht) is that $\mathsf{UC}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{C}_{\mathbb{N}^\mathbb{N}}$, but we have no proof yet.
- Is the Weihrauch lattice a Brouwerian algebra (Heyting lattice)? The answer is "yes" for total Weihrauch reducibility but not known for the ordinary reducibility.
- In a current joint project with Arno Pauly and Stephane Le Roux we are trying to classify the Brouwer Fixed Point Theorem BFT more precisely.
- It is known that $\mathsf{C}_{[0,1]} \equiv_{\mathrm{W}} \mathsf{IVT} \equiv_{\mathrm{W}} \mathsf{BFT}_1 <_{\mathrm{W}} \mathsf{WKL}$, i.e. the one-dimensional Brouwer Fixed Point Theorem is equivalent to the Intermediate Value Theorem and strictly below Weak König's Lemma.
- It is still unclear whether $\mathsf{BFT} \equiv_{\mathrm{W}} \mathsf{WKL}$.
- In this context, one would wish to classify connected closed choice.

Reverse Computable Analysis



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  http://arxiv.org/abs/0905.4679
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  Effective Choice and Boundedness Principles in Computable Analysis (submitted)
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- Vasco Brattka, Matthew de Brecht and Arno Pauly
  Closed Choice and a Uniform Low Basis Theorem (submitted)
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