A Return to Apartness Spaces

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Constructive topology

Two quotes from Errett Bishop:

Very little is left of general topology after that vehicle of classical mathematics has been taken apart and reassembled constructively. With some regret, plus a large measure of relief, we see this flamboyant engine collapse to constructive size. (FCA, 1967)

The problem of finding a suitable constructive framework for general topology is important and elusive. (Letter to dsb, 14 April 1975)

Ten years ago, Luminița Vîță and I began a project to develop a constructive alternative to general topology, based on a primitive notion of "apartness". A book on this has now been submitted to a publisher:

> D.S. Bridges and L.S. Vîță, *Apartness and Uniformity—A Constructive Development*.

Inequality and point-set apartness

Our underlying set X is inhabited and is equipped with a binary relation $\neq$ of **inequality** satisfying

$$\begin{aligned} x \neq y &\Rightarrow \neg (x = y), \\ x \neq y &\Rightarrow y \neq x. \end{aligned}$$

The **complement** of $S \subset X$ is then

$$\sim S = \left\{ x \in X : \forall_{y \in S} \, (x \neq y) \right\}$$

Two other complements of $S \subset X$:

- **logical complement**

$$\neg S = \{x \in X : x \notin S\}$$

- **apartness complement**

$$-S = \{x \in X : x \bowtie S\},$$

where $x \bowtie S$ means that x **is apart from** S.

It will follow from the axioms for $\bowtie$ that

$$-S \subset {\sim}S \subset \neg S.$$

Axioms for a **point-set pre-apartness** on X :

A1 $x \bowtie \emptyset$

A2 $x \bowtie A \Rightarrow x \in {\sim}A$

A3 $x \bowtie A \cup B \Leftrightarrow x \bowtie A \wedge x \bowtie B$

A4 $x \in -A \subset {\sim}B \Rightarrow x \bowtie B$

Canonical examples:

- A quasi-metric space (X, ρ), where

$$x \bowtie S \Leftrightarrow \exists_{r>0} \forall_{y \in S} \left(\rho(x, y) > r \right).$$

- A topological space (X, τ), where

$$x \bowtie A \Leftrightarrow \exists_{U \in \tau} \left(x \in U \subset {\sim}A \right).$$

To turn our pre-apartness into a **point-set apartness** we add axiom

A5 $x \bowtie A \Rightarrow \forall_{y \in X} \left(x \neq y \vee y \bowtie A\right)$

Canonical example: a metric space.

Let Y be an inhabited subset of X. We have a natural inequality $\neq_Y$ and point-set relation $\bowtie_Y$ defined for $y, y' \in Y$ and $S \subset Y$ by

$$\begin{aligned} y \neq_Y y' &\Leftrightarrow y \neq y', \\ y \bowtie_Y S &\Leftrightarrow y \bowtie S \end{aligned}$$

where on the right side, $\neq$ and $\bowtie$ are the original inequality and pre-apartness on X. We say that $\neq_Y$ and $\bowtie_Y$ are **induced** on Y by their counterparts on X.

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It is easy to show that $\bowtie_Y$ satisfies **A1–A3**. If, in addition, it satisfies **A4** in the form

$$(Y - A \subset Y \sim B) \Rightarrow (Y - A \subset Y - B),$$

then it is a pre-apartness on Y; in that case, taken with the induced inequality and pre-apartness, Y is called a **pre-apartness subspace** of X; if $\bowtie_Y$ satisfies **A5**, then we call Y an **apartness subspace** of X. We usually omit the subscript "Y".

Proposition. *Let* $(X, \bowtie)$ *be a pre-apartness space satisfying the condition*

$$\forall_{x,y \in X} \forall_{S \subset X} \left((x \in -S \wedge y \notin -S) \Rightarrow x \neq y \right).$$

Then every inhabited subset of X *is a pre-apartness subspace.*

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Then every inhabited subset of X *is a pre-apartness subspace.*

We refer to condition (1) as the **reverse Kolmogorov property (RKP).**

It is equivalent to

$$\forall_{S \subset X} \left(\neg - S = \sim - S \right).$$

If X is an *apartness* space and Y is an inhabited subset of X, then, since **A5** both implies condition the RKP and is inherited from $\bowtie_X$ by $\bowtie_Y$, it follows, with reference to the above proposition, that Y is also an apartness space.

Apartness from Topology

We assume that every topological space (X, τ) comes equipped with an inequality $\neq$.

For a point x and a subset A of such a space we define

$$x \bowtie_\tau A \Leftrightarrow \exists_{U \in \tau} \left(x \in U \subset {\sim}A\right)$$

Thus $x \bowtie_\tau A$ if and only if x belongs to $({\sim}A)^\circ$.

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The pair $(X, \bowtie_{\tau})$ is a **topological pre-apartness space**.

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On a quasi-metric apartness space the pre-apartness corresponding to the quasi-metric topology coincides with the quasi-metric apartness.

If $\bowtie_\tau$ satisfies **A5**, then $(X, \bowtie_\tau)$ is a **topological apartness space**.

If (X, τ) has the **topological A5 property,**

$$\forall_{x \in X} \forall_{U \in \tau} \left(x \in U \Rightarrow \forall_{y \in X} (x \neq y \vee y \in U) \right),$$

then $\bowtie_\tau$ satisfies

A5 $x \bowtie A \Rightarrow \forall_{y \in X} \ (x \neq y \vee \ y \bowtie A)$

Recall the reverse Kolmogorov property

RKP $\forall_{x,y\in X}\forall_{S\subset X}\left((x\in -S\wedge y\notin -S)\Rightarrow x\neq y\right).$

We introduce the **topological reverse Kolmogorov property (tRKP)** on a topological space (X,τ):

$$\forall_{x,y\in X}\forall_{U\in\tau}\left((x\in U\wedge y\notin U)\Rightarrow x\neq y\right).$$

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This condition

- always holds classically for the denial inequality, and

- is a constructive consequence of the topological **A5** property.

Thus it holds if the inequality on X is discrete, and in any quasi-metric space.

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Proposition. *Let* (X, τ) *be a topological space with the tRKP, let* Y *be a subspace of* X, *and let* τ_Y *be the subspace topology induced on* Y *by* τ. *Then* (Y, τ_Y) *also has the tRKP.*

Topology from Apartness

Let $(X, \bowtie)$ be a pre-apartness space.

A subset S of X is **nearly open** if it is a union of apartness complements.

The nearly open sets form the **apartness topology** on X.

Proposition. *In a topological pre-apartness space, every nearly open set is open.*

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Corollary. *If $(X, \bowtie)$ is a pre-apartness space, and τ the corresponding apartness topology. Then $\bowtie_\tau = \bowtie$.*

Proposition. *If* $(X, \bowtie)$ *is a pre-apartness space with the RKP, then* $(X, \tau_{\bowtie})$ *has the tRKP.*

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A topological pre-apartness space (X, τ) is **topologically consistent** if every τ-open subset of X is nearly open.

Every metric apartness space is topologically consistent.

Jeremy Clarke has shown that if every topological apartness space is topologically consistent, then the law of excluded middle holds.

Putting together propositions proved above, we obtain

Proposition. *The following conditions are equivalent on a topologically consistent topological space* (X, τ).

(i) (X, τ) *has the tRKP.*

(ii) $(X, \bowtie_\tau)$ *has the RKP.*

A topological space (X, τ) is **topologically locally decomposable (tLD)** if

$$\forall_{x \in X} \forall_{U \in \tau} \left(x \in U \Rightarrow \exists_{V \in \tau} \left(x \in V \wedge X = U \cup {\sim}V \right) \right).$$

This condition holds classically for any topological space with the denial inequality: just take $V = U$.

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This condition holds classically for any topological space with the denial inequality: just take $V = U$.

Every metric space (X, ρ) is topologically locally decomposable: for if $x \in U$ and U is open in X, then, choosing $r > 0$ such that $B(x, 2r) \subset U$, we can take $V = B(x, r)$.

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Jeremy Clarke's result, taken with this proposition, shows that if every topological apartness space is topologically locally decomposable, then the law of excluded middle holds.

A pre-apartness space $(X, \bowtie)$ is **locally decomposable** if

$$\forall_{x \in X} \forall_{S \subset X} \left(x \in -S \Rightarrow \exists_{T \subset X} \left(x \in -T \wedge X = -S \cup T \right) \right).$$

Local decomposability always holds classically: for if $x \in -S$, then, taking $T = \sim - S$, we have $X = -S \cup T$; also,

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so $x \in -T$.

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Every metric space (X, ρ) is locally decomposable: for if $x \in -S$, then, choosing $r > 0$ such that $B(x, r) \subset -S$, we can take $T = \sim B(x, r/2)$ to obtain $x \in -T$ and $X = -S \cup T$.

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Corollary. *Let* (X, τ) *be a topological pre-apartness space. Then the following conditions are equivalent.*

(i) X *is tLD.*

(ii) X *is topologically consistent, and* $(X, \bowtie_\tau)$ *is LD.*

Types of Continuity

A mapping $f : X \to Y$ between apartness spaces is

- **nearly continuous** if $f(\overline{S}) \subset \overline{f(S)}$ for each $S \subset X$, where the closures are in the apartness topology;

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A mapping $f : X \to Y$ between apartness spaces is

- **nearly continuous** if $f(\overline{S}) \subset \overline{f(S)}$ for each $S \subset X$, where the closures are in the apartness topology;
- **continuous** if $f(x) \bowtie f(S) \Rightarrow x \bowtie S$ for all $x \in X$ and $S \subset X$;
- **topologically continuous** if $f^{-1}(T)$ is nearly open in X for each nearly open $T \subset Y$.

The composition of continuous functions is continuous, and that the restriction of a continuous function to a pre-apartness subspace of its domain is continuous. Analogous remarks hold for nearly continuous functions and for topologically continuous ones.

For a mapping between quasi-metric spaces, continuity in our sense turns out to be equivalent to the usual "ε-δ" property.

Topological continuity implies near continuity.

Proposition. *Let* $(X, \bowtie)$ *be a pre-apartness space such that* $(X, \tau_{\bowtie})$*has the tRKP. Then every topologically continuous mapping of* X *into a pre-apartness space* Y *is continuous.*

In order to obtain a partial converse to the preceding proposition, we introduce the **weak nested neighbourhoods property** for a pre-apartness space X:

$$x \in -A \Rightarrow \exists_{B \subset X}(x \in -B \wedge (\neg B \subset -A)).$$

This property is a simple consequence of local decomposability, and holds in a quasi-metric space.

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Proposition. *Let X be a pre-apartness space, and Y a pre-apartness space with the weak nested neighbourhoods property. Then every continuous function $f : X \to Y$ is topologically continuous.*

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Proposition. *Let X be a pre-apartness space, and Y a pre-apartness space with the weak nested neighbourhoods property. Then every continuous function $f : X \to Y$ is topologically continuous.*

Corollary. *Let f be a mapping of an apartness$^{(*)}$ space X into a pre-apartness space Y with the weak nested neighbourhoods property. Then f is continuous if and only if it is topologically continuous.*

$^{(*)}$ Recall that A5 implies that X has the RKP.

Digression: Quasi-uniform spaces

A **filter base** on a set S is an inhabited set $\mathcal{F}$ of inhabited subsets of S that has the property

F1 The intersection of two sets in $\mathcal{F}$ is in $\mathcal{F}$.

If, in addition, $\mathcal{F}$ has the property

F2 Every subset of S that contains a set in $\mathcal{F}$ is itself in $\mathcal{F}$,

then it is called a **filter** on S.

The set of all neighbourhoods of a given point in a topological space is a filter.

Every filter base $\mathcal{G}$ generates a unique filter whose elements are the supersets of members of $\mathcal{G}$.

Let X be a inhabited set with an inequality $\neq$, and let U, V be subsets of the Cartesian product $X \times X$. We define certain associated subsets as follows:

$$\begin{aligned} U \circ V &= \{(x,y) : \exists_{z \in X}\, ((x,z) \in U \wedge (z,y) \in V)\}, \\ U^1 &= U, \quad U^{n+1} = U \circ U^n \quad (n = 1, 2, \ldots), \\ U^{-1} &= \{(x,y) : (y,x) \in U\}, \\ U[x] &= \{y \in X : (x,y) \in U\}. \end{aligned}$$

We say that U is **symmetric** if $U = U^{-1}$.

Let $\mathcal{U}$ be a family of inhabited subsets of $X \times X$. We say that $\mathcal{U}$ is a **quasi-uniform structure**, or a **quasi-uniformity**, on X if the following conditions hold

U1 $\mathcal{U}$ is a filter on $X \times X$.

U2 For all $x, y \in X$, $x \neq y$ if and only if there exists $U \in \mathcal{U}$ such that $(x, y) \in \neg U$.

U3 For each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V^2 \subset U$.

U4 For each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $X \times X = U \cup \neg V$.

The elements of $\mathcal{U}$ are called the **entourages** of (the quasi-uniform structure on) X, and $(X, \mathcal{U})$—or simply X itself—is called a **quasi-uniform space**.

We call $\mathcal{U}$ a **uniform structure**, and X a **uniform space**, if, in addition to **U1–U4**, for each $U \in \mathcal{U}$ we have $U^{-1} \in \mathcal{U}$; in that case, $U \cap U^{-1}$ is a symmetric entourage.

Note that, since $\neq$ is symmetric, axiom **U2** implies that if $x \neq y$, then there exists $V \in \mathcal{U}$ such that $(y, x) \in \neg V$.

Classically, property **U4** always holds with $V = U$. It is important to postulate it in the constructive theory, since it is the only uniform-space axiom that provides us with alternatives.

(Quasi-)uniform spaces form the natural setting for discussion of uniform continuity, uniform convergence, ...

A quasi-metric space (X, ρ) is a quasi-uniform space in which the quasi-uniformity consists of all subsets of $X \times X$ that contain sets of the form

$$\{(x, y) : \rho(x, y) \leqslant \varepsilon\},$$

for some $\varepsilon > 0$. If ρ is a *metric* on X, then the induced quasi-uniformity is actually a uniform structure.

A quasi-uniform structure $\mathcal{U}$ on X induces a quasi-uniform structure $\mathcal{U}_Y$ on an inhabited subset Y of X: the entourages of $\mathcal{U}_Y$ are the sets $U \cap (Y \times Y)$ with $U \in \mathcal{U}$.

Taken with the inequality and quasi-uniform structure induced by those on X, the set Y is called a **quasi-uniform subspace** of X.

The quasi-uniform structure $\mathcal{U}_Y$ is called the **subspace quasi-uniform structure** on Y.

Proposition. *Let U be an entourage of a quasi-uniform space $(X,\mathcal{U})$. Then for all $x, y \in X$, either $x \neq y$ or $(x, y) \in U$.*

Proof: By **U4**, there exists $V \in \mathcal{U}$ such that $X \times X = U \cup \neg V$. If $(x, y) \in \neg V$, then $x \neq y$, by **U2**.

The **diagonal** of $X \times X$ is the set

$$\Delta = \{(x, x) : x \in X\}.$$

Corollary. *Every entourage of a quasi-uniform space $(X,\mathcal{U})$ contains the diagonal of X.*

Proof: Given $x \in X$ and $U \in \mathcal{U}$, by the preceding proposition we have either $x \neq x$, which is absurd, or $(x, x) \in U$.

Proposition. *If U is an entourage of a quasi-uniform space $(X,\mathcal{U})$, then there exists an entourage V such that $V^2 \subset U$ and $X \times X = U \cup {\sim}V$.*

For each positive integer n we define an n-**chain of entourages** of a quasi-uniform space $(X,\mathcal{U})$ to be an n-tuple $(U_1, \ldots, U_n)$ of entourages such that for each $k \geqslant 2$,

$$U_k^2 \subset U_{k-1} \text{ and } X \times X = U_{k-1} \cup {\sim}U_k.$$

The last proposition ensures that for each $U \in \mathcal{U}$ and each positive integer n there exists an n-chain $(U_1, \ldots, U_n)$ of entourages with $U_1 = U$; when $\mathcal{U}$ is a *uniform* structure, we can also ensure that the entourages U_k are symmetric for each $k \geqslant 2$.

A mapping f from a quasi-uniform space $(X, \mathcal{U})$ to a quasi-uniform space $(Y, \mathcal{V})$ is **uniformly continuous** if

$$\forall_{V \in \mathcal{V}} \exists_{U \in \mathcal{U}} \forall_{x, x' \in X} \left(\left(x, x' \right) \in \mathcal{U} \Rightarrow \left(f(x), f(x') \right) \in V \right).$$

When X and Y are metric spaces, this is equivalent to the usual uniform continuity property:

$$\forall_{\varepsilon > 0} \exists_{\delta > 0} \forall_{x, x' \in X} \left(\rho \left(x, x' \right) < \delta \Rightarrow \rho(f(x), f(x')) < \varepsilon \right).$$

Set-set Apartness

Axioms for a **set-set pre-apartness** $\bowtie$ on a set X with an inequality $\neq$:

B1 $X \bowtie \emptyset$

B2 $S \bowtie T \Rightarrow S \subset {\sim}T$

B3 $R \bowtie (S \cup T) \Leftrightarrow R \bowtie S \wedge R \bowtie T$

B4 $S \bowtie T \Rightarrow T \bowtie S$

Then

$$x \bowtie S \overset{\text{def}}{\iff} \{x\} \bowtie S$$

defines the corresponding point-set pre-apartness.

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To turn our pre-apartness into a **set-set apartness** we add the **axiom of local decomposability:**

B5 $x \in -S \Rightarrow \exists_T (x \in -T \wedge \forall_y (y \in -S \vee y \in T))$

If **B5** holds, then the above point-set pre-apartness is a point-set apartness.

The canonical example of a set-set apartness space is a quasi-uniform space $(X,\mathcal{U})$ with the **quasi-uniform pre-apartness** defined by

$$S \bowtie T \Longleftrightarrow \exists_{U \in \mathcal{U}} \left(S \times T \subset \sim U \right).$$

A special case of this is when $\mathcal{U}$ arises from a metric ρ on X; the metric apartness on X is then given by

$$S \bowtie T \Longleftrightarrow \exists_{\varepsilon > 0} \forall_{s \in S} \forall_{t \in T} \left(\rho(s,t) \geqslant \varepsilon \right).$$

Does every apartness arise from a uniform structure?

Classically: *yes* if the apartness satisfies the **Efremovič condition**

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Constructively: not even in that case. To see this, we prove an amusing

Lemma. *If there exists a pre-apartness space X such that*

$$A \bowtie B \Rightarrow \forall_{x \in X} \left(x \notin A \vee x \notin B \right)$$

and any two disjoint subsets of a singleton are apart—that is, if $x \in X$, $A \subset \{x\}$, $B \subset \{x\}$, and $A \cap B = \varnothing$, then $A \bowtie B$— then the weak law of excluded middle

$$\neg P \vee \neg\neg P$$

holds.

Proof: Suppose there exists such a pre-apartness space X. Fixing $\xi \in X$, consider any statement P, and define

$$\begin{aligned} A &= \{x \in X : x = \xi \wedge P\}, \\ B &= \{x \in X : x = \xi \wedge \neg P\}. \end{aligned}$$

Then A and B are disjoint subsets of $\{\xi\}$, so, by our second hypothesis, $A \bowtie B$.

By our first hypothesis, either $\xi \notin A$, in which case $\neg P$ holds; or else $\xi \notin B$ and therefore $\neg\neg P$ holds.

The somewhat eccentric second hypothesis in this lemma holds classically for any pre-apartness space X.

It holds constructively if the apartness on X is induced by a quasi-uniform structure $\mathcal{U}$: for if $\xi \in X$, and A, B are disjoint subsets of $\{\xi\}$, then, taking any $U \in \mathcal{U}$, we have $A \times B = \varnothing \subset {\sim}U$; whence $A \bowtie B$.

Proposition. *Let X be an inhabited set with the denial inequality. Then*

$$A \bowtie_0 B \Leftrightarrow (A = \varnothing \vee B = \varnothing)$$

defines a set-set apartness on X that satisfies the Efremovič condition. If this apartness is induced by a uniform structure, then the weak law of excluded middle holds.

Strong and Uniform Continuity

A mapping f between apartness spaces X, Y is **strongly continuous** if

$$f(A) \bowtie f(B) \Rightarrow A \bowtie B.$$

For uniform apartness spaces, uniform continuity implies strong continuity. What about the converse?

We say that two sequences $(x_n)_{n\geqslant 1}$, $(x'_n)_{n\geqslant 1}$ in a uniform space $(X, \mathcal{U})$ are **eventually close** if

$$\forall_{U\in\mathcal{U}}\exists_N\forall_{n\geqslant N}\left(\left(x_n, x'_n\right)\in U\right).$$

A mapping f of X into a uniform space Y is **uniformly sequentially continuous** if the sequences $(f(x_n))_{n\geqslant 1}$, $(f(x'_n))_{n\geqslant 1}$ are eventually close in Y whenever $(x_n)_{n\geqslant 1}$, $(x'_n)_{n\geqslant 1}$ are eventually close in X—in other words, if the input sequences are eventually close, then so are the output ones.

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Theorem. *For mappings between uniform apartness spaces, strong continuity implies uniform sequential continuity.*

The proof is long and involved, and uses a new general technique that has several applications.

Working classically, we can always upgrade the conclusion of this theorem to uniform continuity. But in the general case we cannot do this constructively: the statement

"Every strongly continuous mapping between uniform spaces is uniformly continuous"

is equivalent, over **BISH**, to Ishihara's principle **BD**-$\mathbf{N}$, which is known to be independent of **BISH**.

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However, when the range of the mapping has a certain important property, then we can carry through the upgrade.

A uniform space $(X,\mathcal{U})$ is **totally bounded** if for each $U \in \mathcal{U}$ there exist finitely many points $x_1, \ldots, x_n$ of X such that

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When X is a metric space, this definition of total boundedness is equivalent to the usual one.

Theorem. *Let f be a strongly continuous mapping of a uniform space onto a totally bounded uniform space. Then f is uniformly continuous.*

Proximal convergence

Let X be an inhabited set, and $(Y, \bowtie)$ a set-set pre-apartness space.

A sequence $(f_n)_{n \geqslant 1}$ of elements of Y^X is **proximally convergent** to f in Y^X if

$$\forall_{A \subset X} \forall_{B \subset Y} \left(f(A) \bowtie B \Rightarrow \exists_N \forall_{n \geqslant N} \left(f_n(A) \bowtie B \right) \right)$$

When the pre-apartness on Y is induced by a uniform structure $\mathcal{U}$, we have two other notions of convergence for $(f_n)_{n\geqslant 1}$ to f in Y^X:

- $\triangleright$ **uniform sequential convergence,** in which for each sequence $(x_n)_{n\geqslant 1}$ in X, the sequences $(f_n(x_n))_{n\geqslant 1}$ and $(f(x_n))_{n\geqslant 1}$ are eventually close;

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$$\forall_{U \in \mathcal{U}} \exists_N \forall_{n \geqslant N} \forall_{x \in X} \left((f_n(x), f(x)) \in U \right).$$

Uniform convergence implies (i) uniform sequential convergence and (ii) proximal convergence.

Are proximal and uniform convergence equivalent for sequences of functions?

The main results—and their tricky proofs—are analogous to those we found when looking at strong, uniform sequential, and uniform continuity.

Theorem. *Let* X *be a inhabited set,* Y *a uniform space, and* $(f_n)_{n \geqslant 1}$ *a sequence in* Y^X *that converges proximally to* $f \in Y^X$. *Then* $(f_n)_{n \geqslant 1}$ *is uniformly sequentially convergent to* f.

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Theorem. *Let X be an inhabited set, Y a totally bounded uniform space, and $(f_n)_{n \geqslant 1}$ a sequence in Y^X that converges proximally to $f \in Y^X$. Then $(f_n)_{n \geqslant 1}$ is uniformly convergent to f.*

Untouched Topics

Topics untouched in these lectures but of considerable significance, and often technical difficulty, in the theory of apartness and uniformity include

Apartness analogues of compactness and connectedness

Product and quotient apartness spaces.

An axiomatic theory of apartness on abstract lattices.

Clarification of the connections between this project (especially the part on lattice apartness) and that of the groups working on formal topology and on aspects of topology related to computer science.

Two Research Monograph References

D.S. Bridges and L.S. Vîță, *Techniques of Constructive Analysis*, Universitext, Springer Verlag, Heidelberg, 2006.

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