A **Galois connection** between **basic covers** and **binary positivities**

Francesco Ciraulo
(j.w.w. Giovanni Sambin)

Department of Mathematics and Computer Science
University of PALERMO (Italy)
ciraulo@math.unipa.it
www.math.unipa.it/∼ciraulo

Workshop on Constructive Aspects of Logic and Mathematics
Kanazawa, 8 - 12 March 2010

Overview

1. **BASIC topologies**
   - basic covers = closure operators = saturations
   - binary positivities = interior operators = reductions
   - "compatibility"

Overview

1. **BASIC topologies**
   - basic covers = closure operators = saturations
   - binary positivities = interior operators = reductions
   - "compatibility"
2. **DENSE** and **THICK** basic topologies
   - dense ≈ representable (= pointwise definable)
   - thick ≈ generated (inductively-coinductively)

Overview

1. **BASIC topologies**
   - basic covers = closure operators = saturations
   - binary positivities = interior operators = reductions
   - "compatibility"
2. **DENSE** and **THICK** basic topologies
   - dense $\approx$ representable (= pointwise definable)
   - thick $\approx$ generated (inductively-coinductively)
3. **Properties**
   - **Galois** connection between covers and positivities
   - **Dense** and **thick** as **(co)reflections**
   - **Adjunction** between dense and thick

Part I

Basic topologies

Basic covers

a COVER relation over a set S is

an (infinitary) relation $\lhd$ between elements $a, b, \ldots \in S$ and subsets $U, V, \ldots \subseteq S$, such that:

$$\underbrace{\frac{a \in U}{a \lhd U} \qquad \frac{a \lhd U \quad \forall u (u \in U \Rightarrow u \lhd V)}{a \lhd V}}_{BASIC\ \ COVER} \qquad \frac{a \lhd U \quad a \lhd V}{a \lhd U \ldots V} \qquad \ldots$$

Saturations (closure operators)

Basic cover $\simeq$ saturation (closure operator)

$$\begin{array}{ccc} a \lhd U & \leftrightsquigarrow & \mathcal{A}(U) \\ a \in \mathcal{A}(U) & & \{a \in S \mid a \lhd U\} \end{array}$$

a SATURATION is

$$\mathcal{A} : \mathcal{P}(S) \to \mathcal{P}(S)$$

$$U \subseteq \mathcal{A}(U) \qquad \mathcal{A}\mathcal{A} = \mathcal{A} \qquad \frac{U \subseteq V}{\mathcal{A}(U) \subseteq \mathcal{A}(V)}$$

Binary positivities and Reductions (interior operators)

$$\ltimes : S \times \mathcal{P}(S) \to Prop$$

$$a \ltimes U$$

$$\{a \in S \mid a \ltimes U\}$$

$$\frac{a \ltimes U}{a \in U}$$

$$\frac{a \ltimes U \quad \forall v(v \ltimes U \Rightarrow v \in V)}{a \ltimes V}$$

$$\mathcal{J} : \mathcal{P}(S) \to \mathcal{P}(S)$$

$$a \,\epsilon\, \mathcal{J}(U)$$

$$\mathcal{J}(U)$$

$$\left.\begin{array}{c} \mathcal{J}(U) \subseteq U \\ \\ \dfrac{\mathcal{J}(U) \subseteq V}{\mathcal{J}(U) \subseteq \mathcal{J}(V)} \end{array}\right\} \qquad \boxed{\begin{array}{c} \mathcal{J}(U) \subseteq U \\ \mathcal{J}\mathcal{J} = \mathcal{J} \\ \dfrac{U \subseteq V}{\mathcal{J}(U) \subseteq \mathcal{J}(V)} \end{array}}$$

Basic (formal) topologies

Covers and positivities linked by COMPATIBILITY
(saturations) (reductions)

a **BASIC TOPOLOGY** *is*

$(S , \mathcal{A} , \mathcal{J})$ $(S , \lhd , \ltimes)$

$(set, \ saturation, \ reduction)$ $(set, \ cover, \ positivity)$

$+$

compatibility :

$$\frac{\mathcal{A}(U) \between \mathcal{J}(V)}{U \between \mathcal{J}(V)}\ [1] \qquad \frac{a \lhd U \quad a \ltimes V}{(\exists u \in U)(u \ltimes V)}$$

[1] Where: $(U \between V) \overset{def}{\Longleftrightarrow} (U \cap V$ is inhabited).

What's next?

Two ways to construct a basic topology:

- inductive-coinductive generation (Martin-Löf & Sambin)
- representation via operators induced by a relation (Basic Picture)

Aim of the talk	:	to generalise these two methods and show that they are "DUAL" (in some precise sense).
Questions we want to address (and to answer)	:	*Is $\ltimes$ always determined by $\lhd$? When does this happen? What about the converse?*

Part II

Thick basic topologies

inductively-coinductively generated
basic topologies

Inductive generation of $\lhd$

Problem : given some (= a set-indexed family of) "axioms"
define the least $\lhd$ satisfying the axioms.

Solution: inductive generation

For any (set-indexed) family of axioms $a \prec U_i$, define $\lhd$ by:

$$\boxed{\frac{a \in V}{a \lhd V}} \quad \& \quad \boxed{\frac{a \prec U_i \quad (\forall x \in U_i)(x \lhd V)}{a \lhd V}} \quad \&$$

$$\boxed{\frac{a \lhd V \quad V \subseteq P \quad \forall i(U_i \subseteq P \Rightarrow a \in P)}{a \in P} \ (\textit{induction})}$$

$\lhd$ is *the least cover* such that:

$$\frac{a \prec U}{a \lhd U}$$

Generating positivity by co-induction
(Martin-Löf & Sambin)

$$\boxed{\frac{a \ltimes V}{a \in V}} \quad \& \quad \boxed{\frac{a \prec U_i \quad a \ltimes V}{(\exists x \in U_i)(x \ltimes V)}} \quad \&$$

$$\boxed{\frac{a \in P \quad P \subseteq V \quad \forall i(a \in P \Rightarrow U_i \between P)}{a \ltimes V}\ (coinduction)}$$

$\ltimes$ is *the greatest positivity* which is compatible with $\lhd$

$$\boxed{\frac{a \in V}{a \lhd V}} \quad \boxed{\frac{a \prec U_i \quad (\forall x \in U_i)(x \lhd V)}{a \lhd V}} \quad \boxed{\frac{a \lhd V \quad V \subseteq P \quad \forall i(U_i \subseteq P \Rightarrow a \in P)}{a \in P}}$$

. . . with operators. . .

$$\begin{array}{rcl} \text{(Set-indexed) family of axioms} & \rightsquigarrow & \text{Operator } \mathcal{F} : \mathcal{P}(S) \to \mathcal{P}(S) \\ a \prec U & \rightsquigarrow & a \in \mathcal{F}(U) \end{array}$$

$$\mathcal{A}(V) \quad \stackrel{def}{=} \quad \bigcap \left\{ P \;\middle|\; V \subseteq P \text{ and } \frac{X \subseteq P}{\mathcal{F}(X) \subseteq P} \text{ (for all } X) \right\}$$

$\mathcal{A}$ is **the least saturation which contains** $\mathcal{F}$

$$\mathcal{J}(V) \quad = \quad \bigcup \left\{ P \;\middle|\; P \subseteq V \text{ and } \frac{\mathcal{F}(X) \between P}{X \between P} \text{ (for all } X) \right\}$$

$\mathcal{J}$ is **the greatest reduction which is compatible with** $\mathcal{A}$

The greatest reduction compatible with a given saturation
(Impredicatively)

Let $\mathcal{A}$ be a saturation; then:

(impredicative)

$$\mathbf{J}(\mathcal{A})(V) \stackrel{def}{=} \bigcup \left\{ L \;\middle|\; L \subseteq V \text{ and } \frac{\mathcal{A}(X) \between L}{X \between L} \text{ (for all } X) \right\}$$

is *the greatest reduction* compatible with $\mathcal{A}$

Classically: $\mathbf{J}(\mathcal{A})(U) = -\mathcal{A}(-U)$.
Intuitionistically, $-\mathcal{A}-$ is not a reduction.[2]

[2] As for a counterexample, take $\mathcal{A}$ to be the identity operator.

Thick basic topologies

A basic topology $(S, \mathcal{A}, \mathcal{J})$ is:

generated : if $\mathcal{A}$ and $\mathcal{J}$ are generated inductively and coinductively, respectively, by the **same** axioms;

thick : if $\mathcal{J} = \mathbf{J}(\mathcal{A})$.

GENERATED $\Longrightarrow$ *THICK*

Impredicatively the two notions coincide.

One saturation, many reductions...

$$(\ S\ ,\ \mathcal{A}\ ,\ \ldots\ )$$

Problem: *In how many ways can we fill the spaces?*

The greatest solution: $\mathbf{J}(\mathcal{A})$

The least solution: $\bot$, where $\bot(V) = \emptyset$

$$\text{Each } \mathcal{J}\text{, a different } \underbrace{\textit{class of formal points}}_{\textit{subspace}}.$$

$$\underbrace{(S, \mathcal{A}, \bot)}_{\textit{no points}} \quad \underbrace{(S, \mathcal{A}, \cdots) \quad (S, \mathcal{A}, \cdots) \quad (S, \mathcal{A}, \cdots)}_{\textit{proper subspaces}} \quad \underbrace{(S, \mathcal{A}, \mathbf{J}(\mathcal{A}))}_{\textit{the whole space}}$$

Part III

Dense basic topologies

When points form a set.

Representable basic topologies
(see Sambin's Basic Picture. . .)

Another way to construct a (basic) topology:

Idea : given $\alpha_i \subseteq S (i \in I)$ (the future formal points) define a basic topology "pointwise":

$$a \lhd U \quad \overset{def}{\Leftrightarrow} \quad (\forall i \in I)(a \in \alpha_i \ \Rightarrow \ \alpha_i \between U)$$
$$a \ltimes U \quad \overset{def}{\Leftrightarrow} \quad (\exists i \in I)(a \in \alpha_i \ \& \ \alpha_i \subseteq U)$$

Intuitively[3] : this is the **least** basic topology which has the α_i's as points.

[3] The exact statement require some extra condition.

Sambin's Basic Pairs

Start from $\Vdash : X \times S \to Prop$ (an arbitrary binary **relation**) and define the following four operators:

$$
\begin{array}{rcl}
\mathcal{P}(X) & \rightleftarrows & \mathcal{P}(S) \\
\hline
\{x \mid (\exists u)(u \in U \ \ \& \ \ x \Vdash u)\} = \mathrm{ext}\,(U) & \longleftarrow & U \\
D & \rightarrowtail & \Diamond D = \{a \mid (\exists d)(d \in D \ \ \& \ \ d \Vdash a)\} \\
\{x \mid (\forall u)(x \Vdash u \ \Rightarrow \ u \in U)\} = \mathrm{rest}\,(U) & \longleftarrow & U \\
D & \rightarrowtail & \Box D = \{a \mid (\forall d)(d \Vdash a \ \Rightarrow \ d \in D)\}
\end{array}
$$

Since $\mathrm{ext} \dashv \Box$ and $\Diamond \dashv \mathrm{rest}$ we get that $\mathcal{A} = \Box\,\mathrm{ext}$ is a saturation (the monade of the former adjunction) and $\mathcal{J} = \Diamond\,\mathrm{rest}$ is a reduction (the comonade of the latter adjunction). Moreover, they are compatible, that is: $(S, \mathcal{A}, J) = (S, \Box\,\mathrm{ext}\,, \Diamond\,\mathrm{rest}\,)$ is a basic topology (*represented* by the relation $\Vdash$).

Equivalenlty. . .

$$a \in \mathcal{A}(U) \;\Leftrightarrow\; a \in \Box(\operatorname{ext}(U)) \;\Leftrightarrow\; (\forall x \in X)(a \in \Diamond\{x\} \;\Rightarrow\; \Diamond\{x\} \between U)$$

$$a \in \mathcal{J}(U) \;\Leftrightarrow\; a \in \Diamond(\operatorname{rest}(U)) \;\Leftrightarrow\; (\exists x \in X)(a \in \Diamond\{x\} \;\&\; \Diamond\{x\} \subseteq U)$$

So, given a set-indexed family $\overbrace{\{F_i \subseteq S \mid i \in I\}}^{\{\Diamond\{x\}\subseteq S \mid x\in X\}}$, define:

$$a \in \mathcal{A}(U) \quad \overset{def}{\Leftrightarrow} \quad (\forall i \in I)(a \in F_i \;\Rightarrow\; F_i \between U)$$

$$a \in \mathcal{J}(U) \quad \overset{def}{\Leftrightarrow} \quad (\exists i \in I)(a \in F_i \;\&\; F_i \subseteq U)$$

Properties:

- $(S, \mathcal{A}, J)$ is a basic topology ($\mathcal{A}$ and $\mathcal{J}$ are compatible)
- $\mathcal{J}$ is *the least reduction* s.t. each F_i is **formal closed** ($= \mathcal{J}$-fixed)
- $\mathcal{A}$ is *the greatest saturation compatible with* $\mathcal{J}$

The greatest cover compatible with a given positivity
(Impredicatively)

Impredicatively, every $\mathcal{J}$ is "representable": take the family $\{\mathcal{J}(Z) \mid Z \subseteq S\}$ and check:

$$a \in \mathcal{J}(U) \;\Leftrightarrow\; (\exists Z \subseteq S)(a \in \mathcal{J}(Z) \;\&\; \mathcal{J}(Z) \subseteq U\}$$

(impredicative)

For any reduction $\mathcal{J}$ there exists the greatest saturation compatible with it:

$$a \in \mathbf{A}(\mathcal{J})(U) \quad \overset{def}{\Leftrightarrow} \quad (\forall Z \subseteq S)(a \in \mathcal{J}Z \;\Rightarrow U \between \mathcal{J}Z)$$

Classically: $\mathbf{A}(\mathcal{J})(U) = -\mathcal{J}(-U)$.
Intuitionistically: $-\mathcal{J}-$ is a saturation, BUT not compatible with $\mathcal{J}$.

Dense basic topologies

Definition

A basic topology $(S, \mathcal{A}, \mathcal{J})$ is **dense** if $\boxed{\mathcal{A} = \mathbf{A}(\mathcal{J})}$.

Proposition

$$\textit{REPRESENTABLE} \quad \Longrightarrow \quad \textit{DENSE}$$

(and they coincide impredicatively)

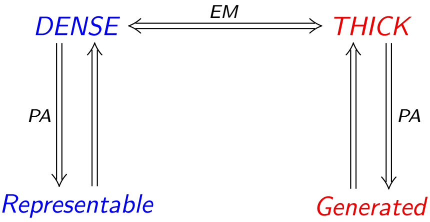
Summing up

$\mathbf{J}(\mathcal{A})$ the greatest reduction compatible with $\mathcal{A}$	$\mathbf{A}(\mathcal{J})$ the greatest saturation compatible with $\mathcal{J}$

IMPREDICATIVELY: they ALWAYS exist!

Predicatively, they exist if...

...if $\mathcal{A}$ is generated (by axioms)	...if the $\mathcal{J}$-closed subsets are set-based (obtained by union from a set-indexed subfamily)



$$\frac{\textit{Dense (Representable)} \quad \Longleftarrow \quad \textit{Generated}}{\textit{Markov}}$$

$$\textit{Representable} \quad \overset{?}{\Longrightarrow} \quad \textit{Thick (Generated)}$$

Part IV

The Galois connection

. . . between saturations and reductions
(on the same set S)

TFAE

- $\mathcal{A}$ and $\mathcal{J}$ are compatible
- $\mathcal{A} \subseteq \mathbf{A}(\mathcal{J})$
- $\mathcal{J} \subseteq \mathbf{J}(\mathcal{A})$

$$\mathcal{J} \subseteq \mathbf{J}(\mathcal{A}) \quad \Longleftrightarrow \quad \mathcal{A} \subseteq \mathbf{A}(\mathcal{J})$$

$$\{\text{reductions over } S\} \underset{\mathbf{J}}{\overset{\mathbf{A}}{\rightleftarrows}} \{\text{saturations over } S\}$$

is a (antitone) Galois connection
(or dual adjunction)

For any basic topology $\mathcal{S} = (S, \mathcal{A}, \mathcal{J})$ we put:

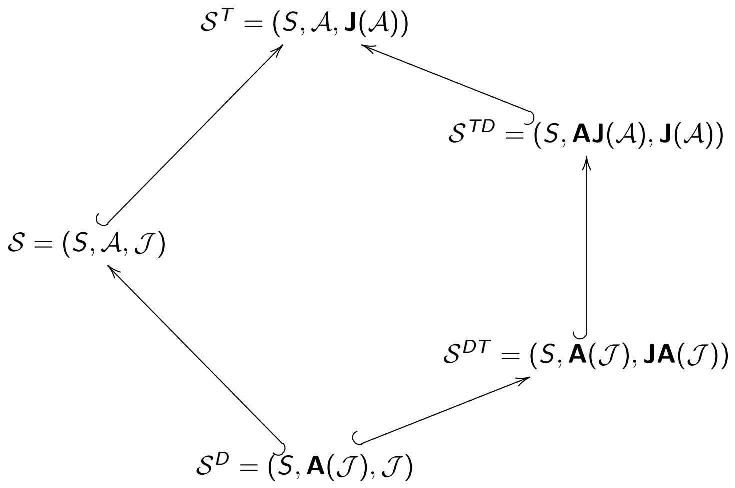
("densification") $\mathcal{S}^D = (S, \mathbf{A}(\mathcal{J}), \mathcal{J})$

the greatest dense (representable) basic topology which is contained in $\mathcal{S}$

("thickification") $\mathcal{S}^T = (S, \mathcal{A}, \mathbf{J}(\mathcal{A}))$

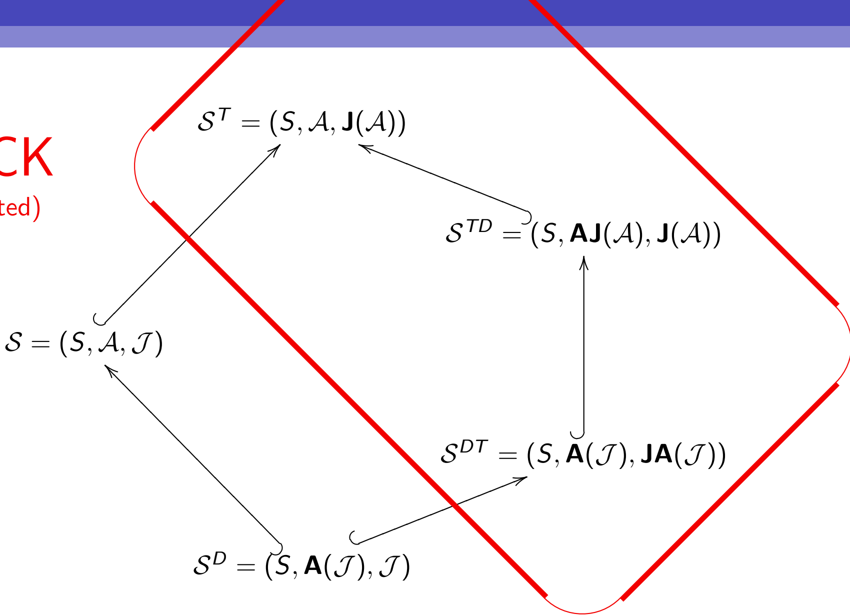
the least thick (generated) basic topology which contains $\mathcal{S}$

$$\mathcal{S}^D \hookrightarrow \mathcal{S} \hookrightarrow \mathcal{S}^T$$

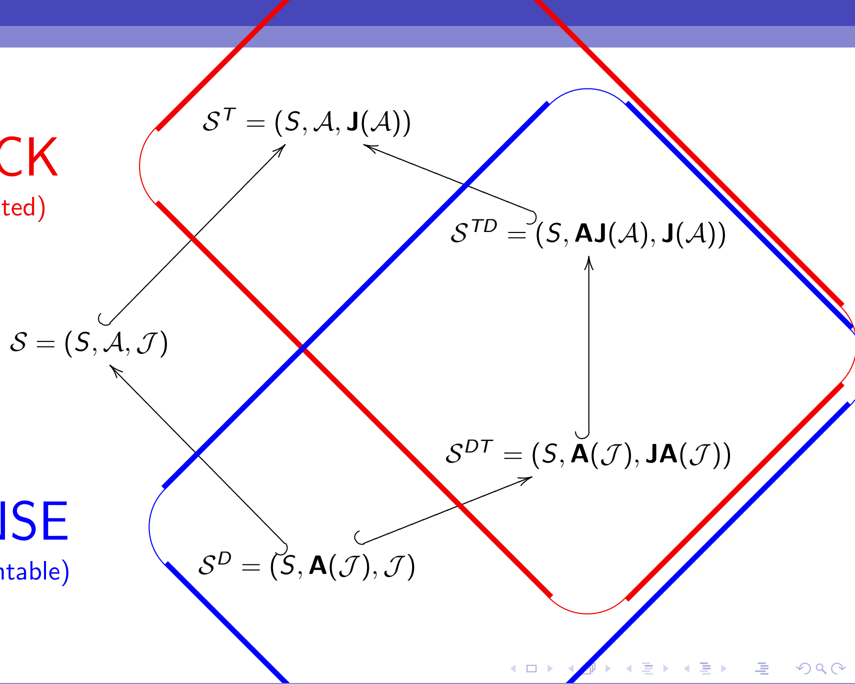

$\mathcal{S}^T = (S, \mathcal{A}, \mathbf{J}(\mathcal{A}))$
$\mathcal{S}^{TD} = (S, \mathbf{AJ}(\mathcal{A}), \mathbf{J}(\mathcal{A}))$
$\mathcal{S} = (S, \mathcal{A}, \mathcal{J})$
$\mathcal{S}^{DT} = (S, \mathbf{A}(\mathcal{J}), \mathbf{JA}(\mathcal{J}))$
$\mathcal{S}^D = (S, \mathbf{A}(\mathcal{J}), \mathcal{J})$

THICK

(generated)



THICK
(generated)



DENSE
(representable)

Part V

Continuity for dense and thick topologies

Continuous relations
(morphisms of basic topologies)

A morphism $(S_1, \mathcal{A}_1, \mathcal{J}_1) \xrightarrow{r} (S_2, \mathcal{A}_2, \mathcal{J}_2)$ is given by a binary relation r between S_1 and S_2 such that:

(covers/saturations)	(positivities/reductions)
$\dfrac{b \lhd_2 V}{r^-[\{b\}] \lhd_1 r^-[V]}$ $\Updownarrow$ $r^-[\mathcal{A}_2(U)] \subseteq \mathcal{A}_1(r^-[V])$	$r[\mathcal{J}_1(U)] \subseteq \mathcal{J}_2(r[U])$
Where: $a \in r^-[V] \overset{def}{\Leftrightarrow} (\exists v \in V)(a\ r\ v)$ (inverse image of r)	Where: $b \in r[U] \overset{def}{\Leftrightarrow} (\exists u \in U)(u\ r\ b)$ (direct image of r)

Morphisms of dense and thick topologies

$\underbrace{(S_1, \mathbf{A}(\mathcal{J}_1), \mathcal{J}_1)}_{(dense)} \xrightarrow{r} (S_2, \mathcal{A}_2, \mathcal{J}_2)$	$(S_1, \mathcal{A}_1, \mathcal{J}_1) \xrightarrow{r} \underbrace{(S_2, \mathcal{A}_2, \mathbf{J}(\mathcal{A}_2))}_{thick}$
is continuous if and only if	
$r\mathcal{J}_1 \subseteq \mathcal{J}_2 r$	$r^- \mathcal{A}_2 \subseteq \mathcal{A}_1 r^-$

In particular a morphism between two dense / thick topologies **is determined by** the condition about reductions / saturations

Corollary

The **full** subcategory

DBTop **TBTop**

whose objects are all the

DENSE THICK

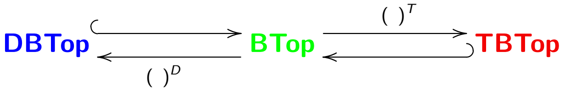
basic topologies

is a

coreflective **reflective**

subcategory of **BTop**

(basic topologies and continuous relations)



$\mathcal{S}^D$ $\mathcal{S}$ $\mathcal{S}^T$

Adjunction between dense and thick

The following two adjunctions. . .

$$\mathbf{DBTop} \underset{(\ )^D}{\overset{\hookrightarrow}{\longleftarrow}} \mathbf{BTop} \underset{\hookleftarrow}{\overset{(\ )^T}{\longrightarrow}} \mathbf{TBTop}$$

. . . yields also the following (by composition). . .

$$\mathbf{DBTop} \underset{(\ )^D{}_{|\mathbf{TBTop}}}{\overset{(\ )^T{}_{|\mathbf{DBTop}}}{\rightleftarrows}} \mathbf{TBTop}$$

References

- F. Ciraulo and G. Sambin, *A Galois connection between closure and interior operators through the notion of basic topology*, in preparation.
- T. Coquand - S. Sadocco - G. Sambin - J. Smith, *Formal topologies on the set of first-order formulae*, J. Symb. Log. **65**, No. 3, 1183-1192 (2000).
- T. Coquand - G. Sambin - J. Smith - S. Valentini, *Inductively generated formal topologies*, Ann. Pure Appl. Logic **124**, No. 1-3, 71-106 (2003).
- P. Martin-Löf, G. Sambin *Generating positivity by coinduction* in G. Sambin, *The Basic Picture: Structures for Constructive Topology*, Oxford University Press, to appear.
- G. Sambin, *The Basic Picture: Structures for Constructive Topology*, Oxford University Press, to appear.

ありがとう

[arigatō]

(Thank you!)