On constructive operational set theory

Laura Crosilla, Leeds University
(Joint work with Andrea Cantini, Florence University)

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Motivation: bridging the gap between Feferman's explicit mathematics and Myhill and Aczel's constructive set theory

The bridge: Constructive Operational Set Theory

- Set–theoretic constructions as union, pairing, exponentiation, are perfectly good operations
  They cannot be represented as objects in set theory as their graphs are too large to be expressed by the set–theoretic notion of function

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  They cannot be represented as objects in set theory as their graphs are too large to be expressed by the set–theoretic notion of function
- We introduce abstract operations as rules vs. functions as set–theoretic graphs

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- Cantini and C. 2008; 2010?; Cantini

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From a constructive point of view, **CZF** has a natural interpretation in Martin–Löf type theory

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- All set–theoretic axioms are **explicit**

The theory **ESTE**

Language: applicative extension, $\mathcal{L}^O$, of the usual first order language of Zermelo-Fraenkel set theory

Language

- $\in, =, \bot, \wedge, \vee, \rightarrow, \exists, \forall$

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- **pair** , **un** , **im** , **sep** , **exp** (set operations)
- $\varnothing$, ω (set constants)

Δ_0 formulas

A formula of $\mathcal{L}^O$ is Δ_0, if
(a) all quantifiers occurring in it, if any, are bounded
(b) it does not contain *App*

Conventions: application terms

(i) Each variable and constant is an application term

(ii) If t, s are application terms then ts is an application term

Abbreviations:

(i) $t \simeq x$ for $t = x$ when t is a variable or constant

(ii) $ts \simeq x$ for $\exists y \exists z\, (t \simeq y \wedge s \simeq z \wedge App(y, z, x))$

(iii) $t\downarrow$ for $\exists x\, (t \simeq x)$

(iv) $t \simeq s$ for $\forall x\, (t \simeq x \leftrightarrow s \simeq x)$

(v) $\varphi(t, \dots)$ for $\exists x\, (t \simeq x \wedge \varphi(x, \dots))$

(vi) $t_1 t_2 \dots t_n$ for $(\dots(t_1 t_2)\dots)t_n$

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For a and b sets or classes, write

- $f : a \to b$ for $\forall x \in a (fx \in b)$,
- $f : \mathbf{V} \to b$ for $\forall x (fx \in b)$
- $f : a^2 \to b$ for $\forall x \in a \forall y \in a (fxy \in b)$
- $f : \mathbf{V}^2 \to b$ for $\forall x \forall y (fxy \in b)$ etc.

Axioms of **ESTE**

ESTE is the following $\mathcal{L}^O$ theory:

- Axioms and rules of first order intuitionistic logic with equality

Extensionality

- $\forall x\,(x \in a \leftrightarrow x \in b) \rightarrow a = b$

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General applicative axioms

- $App(x, y, z) \wedge App(x, y, w) \rightarrow z = w$
- $\mathbf{K}xy = x \wedge \mathbf{S}xy\downarrow \wedge \mathbf{S}xyz \simeq xz(yz)$

Membership operation

- $\mathbf{el} : \mathbf{V}^2 \to \Omega$ and $\mathbf{el}\, xy \simeq \top \leftrightarrow x \in y$

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Set constructors

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- $Ind(\omega) \wedge \forall z\,(Ind(z) \to \omega \subseteq z)$

Some properties of **ESTE**

Lemma

(i) *For each term t, there exists a term $\lambda x.t$ with free variables those of t other than x and such that*

$$\lambda x.t\downarrow \wedge (\lambda x.t)y \simeq t[x := y]$$

(ii) *(Second recursion theorem) There exists a term **rec** with*

$$\mathbf{rec}f\downarrow \wedge (\mathbf{rec}f = e \to ex \simeq fex)$$

Bounded separation

Lemma

For each Δ_0 formula φ with free variables contained in $\{x_1, \ldots, x_k\}$, there is an application term f_φ such that $f_\varphi \downarrow$, $f_\varphi : \mathbf{V}^k \to \Omega$ and

$$f_\varphi\, x_1 \ldots x_k \simeq \top \leftrightarrow \varphi(x_1, \ldots, x_k)$$

Non–extensionality and partiality of operations

Lemma

ESTE *refutes extensionality for operations and totality of application:*

- $\neg[\forall x\,(fx \simeq gx) \rightarrow f = g]$;
- $\neg\forall x\,\forall y\,\exists z\,App(x, y, z)$

Lemma

ESTE *with uniform separation for bounded conditions containing App is inconsistent*

Choice principles for operations

(OAC) $\forall x \in a \exists y \, \varphi(x, y) \to \exists f \forall x \in a \, \varphi(x, fx)$

(GAC) $\forall x \, (\varphi(x) \to \exists y \, \psi(x, y)) \to \exists f \forall x \, (\varphi(x) \to \psi(x, fx))$

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Lemma

(i) **ESTE** + **OAC** *proves* $\varphi \vee \neg\varphi$ *for arbitrary bounded formulas*

(ii) **ESTE** + **GAC** *and* **ESTE** + **GAC** ! *are inconsistent*

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For operations: while **GAC** and **GAC**! are inconsistent, a description operator is consistent with **ESTE** (Cantini)

One can then prove the axiom of unique choice for set–theoretic functions (for Δ_0 formulas)

Proof–theoretic strength

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  We introduce an auxiliary theory **ECST***: Aczel and Rathjen's **ECST** + Exponentiation
  - Reduce **ESTE** to **ECST***: by partial cut–elimination and asymmetric interpretation
  - Reduce **ECST*** to **PA**: the main ingredient is realisability

Extensions by inductive definitions

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Extensions of **CZF** by inductive definitions: Aczel's **REA**, Rathjen's **GID**

Inductive definitions in **CZF**

An inductive definition Φ is a class of ordered pairs
If $(X, a) \in \Phi$, then (X, a) is an inference step of Φ, with set X of premisses and conclusion a

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Note that Γ_Φ is monotone; i.e. for classes Y_1, Y_2,
$Y_1 \subseteq Y_2 \Rightarrow \Gamma_\Phi(Y_1) \subseteq \Gamma_\Phi(Y_2)$

Least fixed points

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Theorem (Aczel) In **CZF**, for any inductive definition Φ there is a smallest Φ–closed class $I(\Phi)$

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(**GID**) If Φ is local and conclusion bounded then $I(\Phi)$ is a set

General inductive definitions in **ESTE**

$MonBd(f, a)$ states that f is monotone and conclusion bounded by a, i.e.

$$\forall x, y\,(x \subseteq y \to \exists z, w \subseteq a\,(z \subseteq w \wedge fx \simeq z \wedge fy \simeq w))$$

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Then (μ) is as follows

$$MonBd(f, a) \to \mu fa \subseteq a \wedge (f(\mu fa) \subseteq \mu fa \wedge \forall y(fy \subseteq y \to \mu fa \subseteq y))$$

Proof theoretic strength

Rathjen: $|\Pi^1_2 - \mathbf{CA} + \mathbf{BR}| \leq |\mathbf{CZF} + \mathbf{GID}| \leq |\Pi^1_2 - \mathbf{CA} + \mathbf{BI}|$

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Cantini: $|\Pi^1_2 - \mathbf{CA}_0| \leq |\mathbf{ESTE} + \mu| \leq |\Pi^1_2 - \mathbf{CA}_0 + \mathbf{BI}|$

What happens in the case of general rather than least fixed points?

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What is the strength of $\mathbf{ESTE} + \hat{\mu}$?

Conjecture: $|\mathbf{ESTE} + \hat{\mu}| = \Gamma_0$