Intuitionistic theorems that fail constructively

Giovanni Curi

gcuri@math.unipd.it

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Intuitionistic theorems...

"Intuitionistic theorems..." = results provable in IZF, or less liberally, in topos logic.

Intuitionistic set theory IZF: a system for set theory with intuitionistic rather than classical logic, but 'as similar as possible' to ZF.

In particular, IZF has

- *Powerset*;
- The full *Separation Scheme:*

$$\{x \in a : \varphi(x, ...)\} \text{ is a set for every } \varphi.$$

Topos logic = Higher-order Heyting arithmetic HHA.

Topos logic has *powertypes*, and *full comprehension* at each type.

... that fail constructively

The constructive Zermelo-Fraenkel set theory CZF is Aczel's formulation of CST.

The system CZF is formulated in the same (first-order) language of ZF, and uses intuitionistic logic in place of classical logic.

$$CZF \subsetneq IZF \subsetneq ZF.$$

Two crucial modifications to the ZF (and IZF) axioms are:

- *The Separation scheme* is weakened to the *Restricted Separation scheme*.

  $\{x \in a : \varphi(x, ...)\}$ is a set *whenever* φ *is bounded*.

  φ is *bounded* if quantifiers in φ appear in the form $\forall x \in b$, $\exists x \in b$.

- The *Powerset axiom* is weakened to the *Subset Collection scheme*.

... that fail constructively (cont.)

The *Subset Collection scheme* is a strengthening of Myhill's *Exponentiation axiom*:

Given any two sets A, B the class B^A of functions from A to B is a set.

$$CZF + EM = IZF + EM = ZF.$$

$$IZF = CZF + Sep + Pow.$$

Remark 1. Although $Pow(X)$ is not a set, it is a *class*, the class of *subsets* of X.

Remark 2. The Dedekind reals R_d form a *set* in CZF (Aczel & Rathjen).

... that fail constructively (cont.)

As shown by various authors (van den Berg & Moerdijk, Lubarsky, Rathjen, Streicher, ...) *CZF is consistent with the following principles*:

Troelstra's principle of uniformity

1. if $(\forall x)(\exists n \in \omega)A(x,n)$, then $(\exists n \in \omega)(\forall x)A(x,n)$.

Every set is subcountable

2. $(\forall x)(\exists U \in Pow(\omega))(\exists f) f : U \twoheadrightarrow x$

... that fail constructively (cont.)

By 1,2 one gets the *Generalized Uniformity Principle*:

GUP For every set a, if $(\forall x)(\exists y \in a)\varphi(x, y)$, then $(\exists y \in a)(\forall x)\varphi(x, y)$.

with which CZF is then consistent. In fact, various extensions of CZF, including

$$\text{CZF+REA+PA+Sep}$$

are consistent with GUP.

Moreover, *constructive type theory* CTT is also consistent with a suitable formulation of this principle (Coquand & Petit).

... that fail constructively (cont.)

A possible *necessary condition* for *constructivity*:

> A precondition to qualify a certain mathematical result as *constructive* is that the result can be derived within some extension of CZF or CTT that is compatible with GUP.

We shall see that e.g. IZF is not one of these extensions.

Remark. Far from being a *sufficient* condition:

$$CZF + Sep\ (\cong 2HA)$$

is compatible with GUP. So the given one is a *very liberal* criterion.

Then "*that fail constructively*" here means "*that cannot be derived within any such extension*".

The Main Lemma

Let CZF* (resp. CTT*) be any extension of CZF (resp. of CTT) that is compatible with GUP.

We shall be working in CST, but corresponding facts also hold for CTT.

Here is our first application of GUP. A (large) $\bigvee$-*semilattice* is a partially ordered class that has suprema for arbitrary *subsets*.

The Main Lemma: *No non-degenerate $\bigvee$-semilattice L can be proved to have a set of elements in CZF*.*

Proof. Assume L is a set. Then, for every set y, the class

$$\{x \in L : 0 \in y\}$$

is a set. Therefore, $(\forall y)(\exists a \in L)a = \bigvee\{x \in L : 0 \in y\}$. In CZF*+GUP one then gets

$$(\exists a \in L)(\forall y)a = \bigvee\{x \in L : 0 \in y\},$$

so that L must be degenerate, as follows by first taking $y = 0$, then $y = \{0\}$. So L is not a set in CZF*+GUP, and thus cannot be proved to be a set in CZF*. ■

The Main Lemma (cont.)

A *class-frame*, or *class-locale*, X is a $\bigvee$-semilattice that has a top element $\top$, binary meets $\wedge$, and that is such that $\wedge$ distributes over $\bigvee$.

Corollary.

1. *No non-degenerate class-frame can be proved to be a set in CZF**.

2. *For* $X \neq \emptyset$, $Pow(X)$ *is a proper class.*

GUP is then inconsistent with IZF.

Remark. With a slightly different proof, the main lemma also holds for class-*preframes*, and, more generally, for *any non-degenerate p.o.-class with a greatest element and joins of directed subsets*. Note that there are dcpo's without a top element that do form a set.

The Main Lemma (cont.)

For every X, Y $\bigvee$-semilattices (resp. preframes),

$$Hom(X, Y)$$

is a '$\bigvee$-semilattice' (resp. a 'preframe'), when ordered pointwise.

It follows that in the categories of $\bigvee$-semilattices and of preframes, no non-singleton hom can be set-indexed.

Luckily, this is not the case for *frames*; we do have examples of $Hom(X, Y)$ that are small.

In particular, **KRegFrm** (classic. dually equivalent to **KHausSp**) is locally small in CZF.

Digression

By the Main Lemma, if we want to have *lattices and complete lattices in the same category*, lattices also should be class-sized (even if they may have sets as carrier).

Moreover, if we want the resulting category to be such that, when considered in IZF, it coincides with the usual category of lattices, something more is needed. One may re-define a *lattice* L to be a class-lattice L with a subclass B s.t.

- *B is a set;*
- *for every $x \in L$, $U_x \equiv (\downarrow x) \cap B$ is a set, and $\bigvee U_x$ (exists and)$= x$.*

Such lattices are, in IZF, exactly the usual lattices. More generally, one could re-define similarly partial orders.

When L is a class-frame, we regain Aczel's definition of a *set-generated class-frame*. As these frames are exactly the usual frames in IZF, *we shall call them simply frames* in the following. These are equivalent to the formal topologies.

Order-completion of $\mathbb{Q}$

The Dedekind reals R_d and the Cauchy reals R_c are *sets* in CZF. Both are not (conditionally) order-complete.

Often in topos theory, and sometimes in constructive mathematics, one also considers the *MacNeille reals* R^* (also called *extended reals*).

R^* is conditionally complete, and in a topos, or in IZF, is a set.

Proposition. *No order-complete extension X of $\mathbb{Q}$ may be proved to form a set in CZF*.*

Proof. Assume X is a set. For every set y, consider the set

$$\{b_1\} \cup \{r \in \mathbb{Q} : r \leq b_2 \ \&\ 0 \in y\},$$

for $b_1 < b_2$ two fixed rational numbers. For every y, this is bounded (by b_2), and inhabited (by b_1), so:

$$(\forall y)(\exists a \in X) a = \bigvee(\{b_1\} \cup \{r \in \mathbb{Q} : r \leq b_2 \ \&\ 0 \in y\}).$$

In CZF*+GUP therefore:

$$(\exists a \in X)(\forall y) a = \bigvee(\{b_1\} \cup \{r \in \mathbb{Q} : r \leq b_2 \ \&\ 0 \in y\}).$$

Taking first $y = 0$, then $y = \{0\}$ we get $b_1 = b_2$, against what we had assumed. ■

Complete lattices, frames, preframes, feature everywhere in mathematics.

As, constructively, they are never carried by sets, several standard constructions assuming implicitly that they are sets need to be reconsidered in CZF, CTT.

Typical examples are lattices obtained by taking various kinds of ideals on a frame:

- compactifications of various types
- the Gleason cover of a locale (and of a topos)
- the proof that every compact regular locale is the retract of a coherent locale
- the completion of a uniform frame
- ...

Some of the results obtained via these constructions, valid in every topos, turn out to be *non-constructive* in the present sense.

The Gleason Cover and the Stone-Čech compactification

Recall that the category of locales and continuous mappings is $\mathbf{Frm}^{op}$, and that a locale is *extremally disconnected* iff $\top_L = a^* \vee a^{**}$ holds for all $a \in L$ ($a^* \equiv a \to \bot$).

> *The Gleason cover of a compact regular locale* is a surjection $e : \gamma L \to L$, with γL a compact, regular and extremally disconnected locale, that is 'minimal' in a certain sense.
>
> In a topos, or in IZF, this cover can be constructed for every compact regular locale (Johnstone).
>
> *The Stone-Čech compactification βL of a locale L* is its compact completely regular reflection.
>
> In a topos, or in IZF, βL exists for every L (Banascewski & Mulvey, Johnstone).

These results are successes of locale theory, as the corresponding facts for spaces require highly non-constructive principles.

We shall prove that, however, both these point-free versions fail constructively, i.e. they are *false* in CZF*+GUP, and therefore not derivable in CZF*.

Stone's Lemma

A classical result of M. Stone states that:

the (compact) frame $Idl(B)$ of ideals of a Boolean algebra B is extremally disconnected if and only if B is complete.

This result also holds in topos logic, and gives the intuitionistic existence of compact extremally disconnected locales.

It is used in the construction of the Gleason cover of a locale L: $\gamma L = Idl(L_{\neg\neg})$, where $L_{\neg\neg}$ is the least dense sublocale of L.

The fact that, for L a frame, $Idl(L)$ is not an admissible construction in CZF*, of course is not enough to conclude that the Gleason cover fails constructively to exist.

We have to show that an object with the properties characterizing the Gleason cover cannot exist.

Failure of Gleason

In contrast with what entailed by Stone's result we have:

Theorem. *No non-degenerate locale L can be proved to be extremally disconnected and compact in CZF*.*

Proof (Sketch). Assume L is extremally disconnected and compact. Then,

$$\forall p \in \Omega,\ \top_L = (\bigvee\{\top_L : 0 \in p\})^* \vee (\bigvee\{\top_L : 0 \in p\})^{**},$$

with $\Omega \equiv \mathrm{Pow}\{0\}$.

By compactness and GUP, there is a non-empty $u_0 = \{x_1, ..., x_n\}$, with

$$u_0 \subseteq \{b \in B_L : b \leq \bigvee\{\top_L : 0 \in p\}^*\} \cup \{b \in B_L : b \leq \bigvee\{\top_L : 0 \in p\}^{**}\}$$

such that $(\forall p \in \Omega)\ \top_L \leq \vee u_0$.

Assuming $\neg(x_1 = \bot_L) \vee ... \vee \neg(x_n = \bot_L)$, one gets $(\forall p \in \Omega)(\neg\neg(0 \in p)\ or\ \neg(0 \in p))$, i.e. [R]DML holds.

Failure of Gleason (cont.)

However, [R]DML is easily seen to be incompatible with GUP, so that

$$\neg(\neg(x_1 = \bot_L) \vee ... \vee \neg(x_n = \bot_L)).$$

This gives $\neg\neg(x_1 = \bot_L)$ &...& $\neg\neg(x_n = \bot_L)$, that is

$$\neg\neg(x_1 = \bot_L \text{ \&...\& } x_n = \bot_L).$$

On the other hand, from $\top_L \leq \vee u_0$ one gets

$$\neg(x_1 = \bot_L \text{ \&...\& } x_n = \bot_L),$$

so that L is not compact and extremally disconnected in CZF*+GUP. ■

We have therefore the following strong refutation of the existence of Gleason covers.

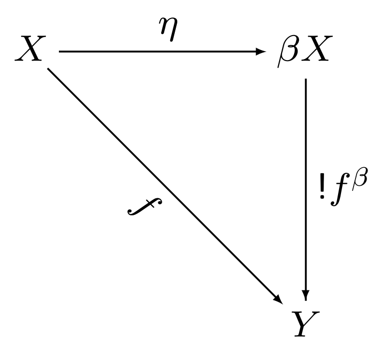
Corollary. *The Gleason cover of a (non-trivial) compact regular locale cannot be defined in CZF*.*

Existence of Stone-Čech compactification

The (generalized) Stone-Čech compactification of a space or locale X is its compact completely regular reflection, i.e., it is a continuous map

$$\eta : X \to \beta X,$$

with βX compact and completely regular, which satisfies the following universal property:



for all compact completely regular Y, and all continuous $f : X \to Y$.

Remark. This universal property gives a bijection $Hom(X, Y) \cong Hom(\beta X, Y)$, for every compact completely regular Y.

Existence of Stone-Čech compactification (cont.)

As said, this compactification exists in a topos, IZF, *for every locale* L.

In CZF+REA we have

Theorem. *The Stone-Čech compactification* βX *of a locale* X *exists if, and only if,* $Hom(X, [0, 1])$ *is a set.*

As, in CZF+sREA+DC, the class $Hom(X, Y)$ is a set whenever X is locally compact and Y is set-presented and regular, in particular we have that

for every locally compact X*,* βX *exists in CZF+sREA+DC.*

Existence of Stone-Čech compactification (cont.)

Thus, in contrast with the Gleason cover, Stone-Čech compactification *does exist* for a class of locales. However,

Theorem. *The Stone-Čech compactification of a non-degenerate Boolean locale X cannot be defined in CZF*.*

In a topos this can be constructed as $Idl(X)$. To prove the theorem we need the following lemmas:

Lemma 1. *A bijection exists between the class of elements of any boolean locale X and $Hom(X, \mathit{Pow}(\{0,1\}))$.*

Proof. To $a \in X$ one associates the map $f_a^- : \mathrm{Pow}(\{0,1\}) \to X$, defined by $f_a^-(\{0\}) = a$, $f_a^-(\{1\}) = a^*$. Conversely, $f : X \to \mathrm{Pow}(\{0,1\})$ defines the open $a_f = f^-(\{0\})$. ■

Existence of Stone-Čech compactification (cont.)

Lemma 2. *Let X be any compact locale. Then the class $Hom(X, Pow(\{0,1\}))$ is a set.*

Idea of the proof. By compactness of L, one can identify the frame homomorphisms from Pow$(\{0,1\})$ to X with a subclass D of the set of mappings $\bar{B}_X^{\{\{0\},\{1\}\}}$, where $\bar{B}_X$ is the range of $\vee$ restricted to Pow$_{fin}(B_X)$.

Then, a mapping $f \in \bar{B}_X^{\{\{0\},\{1\}\}}$ belongs to D iff it satisfies the conditions on frame homomorphims. However, in these hypotheses, such conditions can be given by a bounded formula, and hence D is a set by Restricted Separation.
■

Existence of Stone-Čech compactification (cont.)

Proof of the theorem. Assume X is a Boolean locale.

If βX existed, by Lemma 2, $Hom(\beta X, \mathsf{Pow}(\{0,1\}))$ would be a set in CZF.

Moreover, by the universal property of β,

$$Hom(X, \mathsf{Pow}(\{0,1\})) \cong Hom(\beta X, \mathsf{Pow}(\{0,1\})).$$

Thus $Hom(X, \mathsf{Pow}(\{0,1\}))$ would be a set too. By Lemma 1, X would then be a set in CZF, against the main lemma.

Thus, βX cannot exist in CZF*. ■

Corollary. $Hom(X, \mathcal{R})$, $Hom(X, \mathsf{Pow}([0,1]))$ *are proper classes in CZF*.*

Other failures

- *The compact zero-dimensional reflection of a Boolean locale*.

  One can reason as above, as Pow($\{0,1\}$) is (compact and) zero-dimensional.

- The proof that *any locale is a flat sublocale of a locale of the form* $Idl(D)$*, with* D *a distributive lattice*.

- The fact that any locale is *set-presented* fails for various classes of locales.

As in the preceding cases, the above results hold instead in IZF/HHA.