A constructive look at the Vitali Covering Theorem

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Definition

A Vitali covering of a set $E \subset \mathbb{R}^d$ is a family of closed balls $(B_i)_{i \in I}$ such that for all $x \in E$ and all $\delta > 0$ there exists $i \in I$ with

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Theorem

If $\mathcal{V}$ is a Vitali covering, then there exists disjoint $(B_n)_{n \geqslant 1}$ in $\mathcal{V}$ such that

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Definition

A Vitali covering of a set $[0, 1] \subset \mathbb{R}$ is a family of closed intervals $(B_i)_{i \in \mathbb{N}}$ such that for all $x \in E$ and all $\delta > 0$ there exists $i \in \mathbb{N}$ with

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Theorem

If $\mathcal{V}$ is a Vitali covering, then there exists disjoint $(B_n)_{n \geqslant 1}$ in $\mathcal{V}$ such that for all $\epsilon > 0$ there exists $n \in \mathbb{N}$

$$\sum_{i=1}^{n} |B_i| > 1 - \varepsilon \ .$$

Measure theory

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$$\sum_{i=1}^{\infty} |\ldots| \leqslant c,$$

we do not imply that the series converges, but merely that the partial sums are bounded.

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And the answer is:

No, because there is a recursive counterexample.

A recursive counterexample to VCT

In Russian recursive mathematics there exist α-singular covers of $[0,1]$ (for every $0 < \alpha < 1$). That is a sequence of intervals $(J_n)_{n \geqslant 1}$ (with rational endpoints) such that

- any two J_n are disjoint, or have only an endpoint in common,
- any point belongs to the union of two of these, and
- the partial sums of $\sum_{n \geqslant 1} |J_n|$ are bounded by α.

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(This also shows that the Heine Borel theorem is not provable in RUSS/BISH.)

A recursive counterexample to VCT

Proof.

Take a $\frac{1}{6}$-singular cover $(J_n = [a_n, b_n])_{n \geqslant 1}$.
Triple these in length to $I_n = (2a_n - b_n, 2b_n - a_n)$. Then

- $[0, 1] \subset \bigcup_{n \geqslant 1} I_n$ and
- $\sum_{n \geqslant 1} |I_n| = 3 \sum_{n \geqslant 1} |J_n| \leqslant \frac{1}{2}$

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Let $(I_n^m)_{m\geqslant 1}$ be an enumeration of all intervals with rational endpoints that are a subset of I_n.

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Let $(I_n^m)_{m \geqslant 1}$ be an enumeration of all intervals with rational endpoints that are a subset of I_n. Then $(I_n^m)_{n,m \geqslant 1}$ is a Vitali cover of $[0, 1]$. (Let's call this process "Vitalification".) □

A recursive counterexample to VCT

$\Bigg($ • $\sum_{n\geqslant 1}|I_n| = 3\sum_{n\geqslant 1}|J_n| \leqslant \frac{1}{2}$ $\Bigg)$

Now, if VCT holds, then there exist k, $n_1, \ldots, n_k$ und $m_1, \ldots, m_k$ such that $(I_{n_i}^{m_i})_{i=1}^k$ are pairwise disjoint and

$$\frac{1}{2} < \sum_{i=1}^{k} |I_{n_i}^{m_i}| \leqslant \sum_{i\in\{n_1,\ldots,n_k\}} |I_i| \leqslant \frac{1}{2}.$$

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A contradiction.

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This example is very robust: even adding more assumptions does not seem to help. E.g. The Vitali Cover in the counterexample is totally bounded (using the Hausdorff metric).

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What about other varieties of BISH?

Other frameworks

A good guess is that VCT has something to do with Heine-Borel.

In Simpson's reverse mathematics

$$\mathbf{WWKL} \iff \textit{Vitali Covering Theorem}$$

Bars

We are interested in *bars*, that is sets $B \subset 2^*$ that block every infinite "path".

In symbols:

$$\forall \alpha \in 2^{\mathbb{N}} \exists n \in \mathbb{N} \, (\overline{\alpha} n \in B).$$

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In symbols:

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A bar B is called *uniform* if

$$\exists n \in \mathbb{N} \forall \alpha \in 2^{\mathbb{N}} \exists m \leqslant n \, (\overline{\alpha} m \in B).$$

Bars

FAN and WWKL

Remember the fan theorem

FAN$_\Delta$: *Every decidable bar is uniform.*

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And consider the weaker

WWKL: *For every decidable bar B that is closed under extensions*

$$\lim_{n\to\infty} \frac{|\{u \in B \ : \ |u| = n\}|}{2^n} = 0$$

Bars

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Or equivalently:

WWKL: *For every decidable bar B that is closed under extensions and for every $\epsilon > 0$ there exists N*

$$|\{u \in B \ : \ |u| = N\}| > (1 - \varepsilon)2^N \ .$$

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Trivially,

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The reverse implication seems unlikely to be provable.

Takako Nemoto has shown that

WWKL $\iff$ *Every positive, uniformly continuous function* $f : [0,1] \to \mathbb{R}$ *satisfies the following property: For any* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that* $\mu(\{x : f(x) < \delta\})$ *is defined and*

$$\mu(\{x : f(x) < \delta\}) < \varepsilon.$$

Bars

FAN and WWKL

This is also a nice characterisation:

(For every uniformly continuous map $f : [0,1] \to \mathbb{R}^+$)

$$\textbf{WWKL} \Longleftarrow \textbf{FAN}_\Delta$$

WWKL $\Updownarrow$

$$\forall \epsilon > 0 \exists c > 0$$
$$\mu(\{x \mid f(x) > c\}) > 1 - \epsilon$$

FAN$_\Delta$ $\Updownarrow$

$$\exists c > 0$$
$$f(x) > c$$

another recursive counterexample

(RUSS): Again using a singular cover construct an open cover of the interval with rational endpoints such that

- $[0,1] \subset \bigcup_{n \geqslant 1} I_n$ and
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Now set

$$f(x) = \sum_{i=0}^{\infty} 2^{-n} d(x, -I_n) \ .$$

Then f is uniformly continuous and positively valued.

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But for all $m \in \mathbb{N}$

$$\mu\left(\left\{x \in [0,1] \mid f(x) > 2^{-m}\right\}\right) \leqslant \mu\left(\bigcup_{n=1}^{m+1} I_n\right) \leqslant \sum_{n \geqslant 1} |I_n| \leqslant \frac{1}{2}.$$

Back to VCT

Again, the researches working within Simpson's reverse mathematics have shown that

WWKL $\iff$ *For any covering of* $[0,1]$ *by a sequence of open intervals with rational endpoints* $(I_n)_{n\geqslant 1}$ *we have that for all* $\epsilon > 0$ *there is a* $N \in \mathbb{N}$ *with*

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This is also provable in BISH! (With a slightly different proof).

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Notice also, how this property fails in RUSS.

If we assume **WWKL**, then for any covering of $[0,1]$ by a sequence of open intervals with rational endpoints $(I_n)_{n\geqslant 1}$ we have that for all $\epsilon > 0$ there is a $N \in \mathbb{N}$ with

$$\sum_{n=1}^{N} |I_n| > 1 - \varepsilon \ .$$

With this property it is easy to prove Vitali's Covering Theorem via the Baby Vitali Lemma.

Lemma (Baby Vitali)

Given finitely many intervals with rational endpoints $I_1, \dots, I_n$ *there are finitely many indices* $k_1, \dots, k_m$ *such that* $I_{k_1}, \dots, I_{k_m}$ *are disjoint and*

$$\mu\left(\bigcup_{i=1}^{n} I_i\right) \leqslant 3\mu\left(\bigcup_{i=1}^{m} I_{k_i}\right)$$

Together we can prove:

Lemma

Assuming **WWKL**. *If* $a, b \in \mathbb{Q}$ *are such that* $0 \leqslant a < b \leqslant 1$ *and* $(I_n)_{n \geqslant 1}$ *is a Vitali covering of* $[0, 1]$, *then there exist* $n_1, \ldots, n_k$ *such that*

- $I_{n_1}, \ldots, I_{n_k}$ *are disjoint,*
- $I_{n_i} \subset (a, b)$, *and*
- $\sum_{i=1}^{k} |I_{n_k}| > c(b - a)$.

(For a fixed $\frac{1}{3} > c > 0$*)*

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Iterating this method constructs the desired sequence of VCT, since

$$\lim_{n \to \infty} (1 - c)^n = 0 \ .$$

We can also prove the more general result for arbitrary (not necessarily rational) intervals.

A proof that VCT implies **WWKL**

Remember

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> $$\sum_{n=1}^{N} |I_n| > 1 - \varepsilon \ .$$

Start with such a cover $(I_n)_{n\geqslant 1}$ and let $(I_n^m)_{n,m\geqslant 1}$ be its Vitalification. If VCT holds, then for an arbitrary $\varepsilon > 0$ there exist k, $n_1, \ldots, n_k$ und $m_1, \ldots, m_k$, such that $(I_{n_i}^{m_i})_{i=1}^k$ are pairwise disjoint and

$$1 - \varepsilon < \sum_{i=1}^{k} |I_{n_i}^{m_i}| \leqslant \sum_{i\in\{n_1,\ldots,n_k\}} |I_i| \leqslant \sum_{n=1}^{\max\{n_1,\ldots,n_k\}} |I_n| \ .$$

Formal topology

Anton Hedin has given a proof of Vitali's covering theorem in formal topology. His definition is:

Let $V \subset R$ and $(p, q) \in R$, if

1. $(p, q) \lhd V$, *and*
2. $(r, s) \leq (p, q)$ *implies* $(r, s) \lhd V \cap \{(r, s)\}_{\leq}$

we say that V is a Vitali covering of (p, q).

Furthermore, V is a Vitali covering of $U \subset R$, if V is Vitali covering of every $(p, q) \in U$.

Where $R = \{(p, q) \in \mathbb{Q} \mid p < q\}$.

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- More equivalencies of **WWKL**?

Moral of the story

- VCT is not provable in recursive models of BISH.
- It is equivalent to **WWKL** over BISH. It holds in Brouwer's intuitionism and formal topology.
- The basic constructions are similar in all proofs and counterexamples.
- More equivalencies of **WWKL**?
- Similarities to Brown, Giusto and Simpson's work are not intended and purely incidental.

Thanks

Many thanks to Anton Hedin, Douglas Bridges, Maarten Jordens and especially the organisers.

Questions?