Locally Scott formal topologies and constructive interval analysis

Workshop on Constructive Aspects of Logic and Mathematics
Kanazawa, 8-12 March 2010

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Background

Formal topology provides a constructive framework for the theory of domains.

- (Sambin, 1987): The formal space $\mathsf{Pt}(\mathcal{A})$ of a unary (or Scott) formal topology $\mathcal{A}$ is a Scott domain, and every Scott domain is isomorphic to the formal space of a Scott formal topology.

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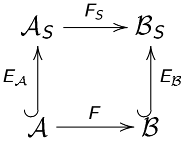
- (Negri, 2002): extended to include the more general notion of continuous domain. The corresponding class of formal topologies being the (stable) locally Scott formal topologies.

Background

- Any formal topology $\mathcal{A}$ continuously embeds in a Scott formal topology $\mathcal{A}_S$ and any morphism $F : \mathcal{A} \to \mathcal{B}$ lifts to a morphism $F_S : \mathcal{A}_S \to \mathcal{B}_S$ such that the following diagram commutes



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$$\begin{array}{ccc} \mathcal{A}_S & \xrightarrow{F_S} & \mathcal{B}_S \\ {\scriptstyle E_{\mathcal{A}}}\uparrow & & \uparrow{\scriptstyle E_{\mathcal{B}}} \\ \mathcal{A} & \xrightarrow{F} & \mathcal{B} \end{array}$$

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- We give a similar embedding and lifting result for the class of locally Scott formal topologies.

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Classically the formal space $\mathrm{Pt}(\mathcal{R}_P)$ corresponds to the interval domain $\mathbf{I}\mathbb{R}$, central in the domain theoretic framework for differential calculus as developed by Edalat et al.
We show that some of the work in this area can be done constructively by using the embedding $\mathcal{R} \hookrightarrow \mathcal{R}_P$ and the extension of morphisms.

Formal spaces

A formal topology $\mathcal{A} = (A, \leqslant, \lhd)$, consists of a pre-ordered set $(A, \leqslant)$ together with a cover relation $\lhd$ satisfying, for all $a \in A$ and $U \subseteq A$

$$\begin{array}{llll} \text{(Ref)} & a \in U \Rightarrow a \lhd U, & \text{(Ext)} & a \leqslant b \Rightarrow a \lhd \{b\}, \\ \text{(Tra)} & a \lhd U \;\&\; U \lhd V \Rightarrow a \lhd V, & \text{(Loc)} & a \lhd U \;\&\; a \lhd V \Rightarrow a \lhd U \wedge V. \end{array}$$

Here $U \wedge V$ denotes the formal intersection of U and V

$$\{x \in X : (\exists u \in U)(\exists v \in V) x \leq u \;\&\; y \leq v\}.$$

$\mathcal{A} = (A, \leqslant, \lhd)$ is called overt if there is a subset $\mathrm{Pos}_{\mathcal{A}} \subseteq A$ satisfying

$$\text{(Mon)} \quad a \lhd U \;\&\; a \in \mathrm{Pos}_{\mathcal{A}} \Rightarrow \exists u \in U \cap \mathrm{Pos}_{\mathcal{A}},$$

$$\text{(Pos)} \quad a \lhd \{a\} \cap \mathrm{Pos}_{\mathcal{A}}.$$

Intuitively $a \in \mathrm{Pos}_{\mathcal{A}}$ iff $\neg(a \lhd \varnothing)$.

Formal spaces

A point of a formal topology $\mathcal{A} = (A, \leqslant, \lhd)$ is a subset $\alpha \subseteq A$ satisfying

(P1) $\exists a \in \alpha$,

(P2) $a, b \in \alpha \Leftrightarrow (\exists c \in \alpha) c \leqslant a \;\&\; c \leqslant b$,

(P3) $a \in \alpha \;\&\; a \lhd U \Rightarrow \exists b \in \alpha \cap U$.

We denote by $\mathrm{Pt}(\mathcal{A})$ the collection of points of $\mathcal{A}$ and call it the formal space of $\mathcal{A}$. Its topology is given by the open neighborhoods $\{a\}^*$, $a \in A$, where for any $U \subseteq A$

$$U^* = \{\alpha \in \mathrm{Pt}(\mathcal{A}) : (\exists u \in U) u \in \alpha\}.$$

We call this the spatial topology on $\mathrm{Pt}(\mathcal{A})$.

Formal spaces

A continuous relation $F : \mathcal{A} \to \mathcal{B}$ between formal topologies $\mathcal{A} = (A, \leqslant_{\mathcal{A}}, \lhd_{\mathcal{B}})$ and $\mathcal{B} = (B, \leqslant_{\mathcal{B}}, \lhd_{\mathcal{B}})$ is a relation $F \subseteq A \times B$ satisfying

(A1) $aFb,\ b \lhd_{\mathcal{B}} V \Rightarrow a \lhd_{\mathcal{A}} F^{-1}[V]$,

(A2) $a \lhd_{\mathcal{A}} U$, xFb for all $x \in U \Rightarrow aFb$

(A3) $A \lhd_{\mathcal{A}} F^{-1}[B]$,

(A4) $aFb,\ aFc \Rightarrow a \lhd_{\mathcal{A}} F^{-1}[b \wedge_{\mathcal{B}} c]$.

Here $F^{-1}[Z] \overset{\text{def}}{=} \{x \in A : (\exists z \in Z) xFz\}$.

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Here $F^{-1}[Z] \overset{\text{def}}{=} \{x \in A : (\exists z \in Z)xFz\}$.

A continuous relation $F : \mathcal{A} \to \mathcal{B}$ induces a continuous function $\mathrm{Pt}(F) : \mathrm{Pt}(\mathcal{A}) \to \mathrm{Pt}(\mathcal{B})$ given by $\mathrm{Pt}(F)(\alpha) = \{b \in B : (\exists a \in \alpha)aFb\}$. The identity $I_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ is given by $aI_{\mathcal{A}}b \Leftrightarrow a \lhd_{\mathcal{A}} \{b\}$ and composition of $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ is given by

$$a(G \circ F)c \Leftrightarrow a \lhd_{\mathcal{A}} F^{-1}[G^{-1}(c)].$$

Let **FTop** be the category of formal topologies and continuous relations.

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A Scott (or unary) formal topology $\mathcal{A}$ satisfies

$$a \lhd U \Rightarrow (\exists u \in U) a \lhd u.$$

In this case $(\mathrm{Pt}(\mathcal{A}), \subseteq)$ is an algebraic dcpo, and if $\mathcal{A}$ is consistently complete, i.e. for all $a, b \in A$

$$(\exists c \in A) c \leqslant a, b \Rightarrow a \wedge b \text{ exists in } A,$$

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Given any $\mathcal{A} = (A, \leqslant, \lhd)$ we define $\mathcal{A}_S = (A, \leqslant_S, \lhd_S)$: Let
$a \leqslant_S b \Leftrightarrow a \lhd \{b\}$ and $a \lhd_S U \Leftrightarrow (\exists u \in U) a \lhd u$.
$\mathcal{A}_S$ is a Scott formal topology and the cover on $\mathcal{A}$ yields an embedding $E_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}_S$

$$a E_{\mathcal{A}} b \Leftrightarrow a \lhd \{b\}.$$

Locally Scott formal topologies

Theorem (Negri, 2002)

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A formal topology $\mathcal{A}$ is locally Scott if there is a map $i : A \to \mathcal{P}(A)$ such that for all $a \in A$ and $U \subseteq A$

(1) $a =_{\mathcal{A}} i(a)$,

(2) $a \lhd U \Rightarrow i(a) \lhd_S i(U)$,

where (2) says $a \lhd U \Rightarrow (\forall b \in i(a))(\exists u \in U)(\exists v \in i(u)) b \lhd v$.
$\mathcal{A}$ is stable in case i also satisfies

(3) $A =_{\mathcal{A}_S} i(A)$,

(4) $i(a) \wedge i(b) \lhd_S i(a \wedge b)$.

Locally Scott formal topologies

A formal topology $\mathcal{A}$ is stable locally Scott if it is a retract of its Scott compactification, i.e. if there is a morphism $R : \mathcal{A}_S \to \mathcal{A}$ such that $R \circ E_{\mathcal{A}} = \mathrm{Id}_{\mathcal{A}}$.
For i satisfying (1)-(4) a retraction R_i is given by
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For any $a \in A$ we have

$$\Uparrow_i a = \{b \in A : aR_i b\} = \mathrm{Pt}(R_i)(\Uparrow a) \in \mathrm{Pt}(\mathcal{A})$$

and for any $\alpha \in \mathrm{Pt}(\mathcal{A})$ the collection $\{\Uparrow a\}_{a \in \alpha}$ is directed and $\alpha = \bigcup_{a \in \alpha} \Uparrow a$.

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The way below relation on $\mathrm{Pt}(\mathcal{A})$ can be characterized as

$$\beta \ll \alpha \Leftrightarrow (\exists a \in \alpha) \beta \subseteq \mathord{\uparrow} a \subseteq \alpha.$$

Hence we see that $\alpha = \bigcup_{\beta \ll \alpha} \beta$, for all $\alpha \in \mathrm{Pt}(\mathcal{A})$.

Locally Scott formal topologies

Suppose $\mathcal{A} = (A, \leqslant, \lhd)$ is equipped with a binary relation $\prec \subseteq A \times A$ such that

(*i*) $a =_{\mathcal{A}} \{b : b \prec a\}$,

(*ii*) $a \prec b \lhd c \Rightarrow (\exists d \prec c) a \lhd d$,

(*iii*) $a \prec b \Rightarrow (\exists c \in A) a \prec c \prec b$.

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(*iii*) $a \prec b \Rightarrow (\exists c \in A) a \prec c \prec b$.

Let $i(a) = \{b : b \prec a\}$, then we define $\mathcal{A}_{LS} = (A, \leqslant_{LS}, \lhd_{LS})$ where $a \leqslant_{LS} b \Leftrightarrow a \lhd \{b\}$ and

$$a \lhd_{LS} U \Leftrightarrow i(a) \lhd_S i(U).$$

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Proposition

If $(\mathcal{A}, \prec)$ satisfies (i)-(iii), then $\mathcal{A}_{LS}$ is locally Scott with respect to $i : a \mapsto \{b : b \prec a\}$. Moreover, $E_{\mathcal{A}}$ is an embedding $\mathcal{A} \to \mathcal{A}_{LS}$.

Locally Scott formal topologies

If $\prec$ satisfies the following two conditions, $\mathcal{A}_{LS}$ will be stable,

(iv) $(\forall a \in A)(\exists b \in a) a \prec b$,

(v) $c \lhd a' \prec a \ \& \ c \lhd b' \prec b \Rightarrow (\exists d', d)(a \lhd d' \prec d \lhd a \wedge b)$.

We may always choose $\prec \subseteq A \times A$ as $a \prec b$ iff $a \lhd \{b\}$, in which case

$$\mathcal{A}_{LS} = \mathcal{A}_S.$$

Locally Scott formal topologies

Lemma

A morphism $F : \mathcal{A} \to \mathcal{B}$ between locally Scott formal topologies must satisfy

$$aFb \Leftrightarrow i_{\mathcal{A}}(a) \subseteq F^{-1} i_{\mathcal{B}}(b).$$

Suppose now, $\mathcal{A}$ and $\mathcal{B}$ are equipped with relations $\prec_{\mathcal{A}}$ and $\prec_{\mathcal{B}}$ satisfying $(i) - (v)$ and let $F : \mathcal{A} \to \mathcal{B}$. We define a relation

$$aF_{LS}b \Leftrightarrow \{x : x \prec_{\mathcal{A}} a\} \subseteq F^{-1}\{y : y \prec_{\mathcal{B}} b\}.$$

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Proposition

The relation F_{LS} is a morphism $\mathcal{A}_{LS} \to \mathcal{B}_{LS}$ making the following diagram commute

$$\begin{array}{ccc} \mathcal{A}_{LS} & \xrightarrow{F_{LS}} & \mathcal{B}_{LS} \\ E_{\mathcal{A}} \uparrow & & \uparrow E_{\mathcal{B}} \\ \mathcal{A} & \xrightarrow{F} & \mathcal{B} \end{array}$$

Locally Scott formal topologies

With $a \prec b \equiv a \lhd \{b\}$ for both $\mathcal{A}$ and $\mathcal{B}$ we have $F_{LS} = F$ and the lifting result is exactly the one for Scott formal topologies, saying that F is also a continuous relation $\mathcal{A}_S \to \mathcal{B}_S$ making the following diagram commute

$$\begin{array}{ccc} \mathcal{A}_S & \xrightarrow{F} & \mathcal{B}_S \\ {\scriptstyle E_{\mathcal{A}}}\uparrow & & \uparrow{\scriptstyle E_{\mathcal{B}}} \\ \mathcal{A} & \xrightarrow{F} & \mathcal{B} \end{array}$$

Locally compact metric spaces

(Palmgren, 2006) establishes a full and faithful functor $\mathcal{M}$ from the category of locally compact metric spaces and continuous functions into the category of formal topologies.
A l.c.m.s. (X, d) is mapped to a formal topology $\mathcal{M}(X)$, whose set of basic opens M_X consist of formal ball symbols $b(x, \delta) \in X \times \mathbb{Q}^+$. These are ordered by inclusion and strict inclusion, respectively,

$$b(x, \delta) \leqslant b(y, \varepsilon) \Leftrightarrow d(x, y) \leq \varepsilon - \delta$$

$$b(x, \delta) \prec b(y, \varepsilon) \Leftrightarrow d(x, y) < \varepsilon - \delta$$

The cover on $\mathcal{M}(X)$ is the least cover satisfying the two axioms

(M1) $p \lhd \{s \in M : s \prec p\}$,

(M2) $M \lhd \{b(x, \delta) : x \in X\}$, for any $\delta \in \mathbb{Q}^+$

The formal space $\mathrm{Pt}(\mathcal{M}(X))$ is a metric completion of (X, d) (wrt a metric m induced by d) and when (X, d) is complete the spaces are isomorphic.

Locally compact metric spaces

Lemma

The strict inclusion $\prec$ between formal ball symbols satisfies (i) – (iv).

E.g. (ii) $(a \prec b \lhd c \Rightarrow (\exists d \prec c) a \lhd d)$ follows from local compactness.

If $a \lhd b \Rightarrow a \leqslant b$ in $\mathcal{M}(X)$ then also (v) is satisfied.
This is the case e.g. when X is a normed vector space. From now on we assume this is the case.

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If $a \lhd b \Rightarrow a \leqslant b$ in $\mathcal{M}(X)$ then also (v) is satisfied.
This is the case e.g. when X is a normed vector space. From now on we assume this is the case.

Elements $\alpha \in \mathrm{Pt}(\mathcal{M}(X)_{LS})$ are of the form

$$\alpha = \bigcup_{b(x,\delta) \in \alpha} \Uparrow b(x, r)$$

where in this case $\Uparrow b(x,\delta) = \{c \in M_X : b(x,\delta) \prec c\}$ (since $a \lhd b \prec c$ implies $a \prec c$).

Points $\Uparrow b(x,\delta)$ correspond to closed sets

$$B[x,\delta] = \{y \in \mathrm{Pt}(\mathcal{M}(X)) : m(x,y) \leq \delta\} \subseteq \mathrm{Pt}(\mathcal{M}(X)),$$

in a precise way.

Locally compact metric spaces

Any $\alpha \in \mathrm{Pt}(\mathcal{M}(X)_{LS})$ defines a closed subspace $[\alpha]$ of $\mathcal{M}(X)$ by letting $[\alpha] = \mathcal{M}(X) \smallsetminus \alpha^{\perp}$, i.e. $[\alpha] = (M_X, \leqslant, \lhd_{[\alpha]})$ where

$$a \lhd_{[\alpha]} U \Leftrightarrow a \lhd\ U \cup \alpha^{\perp}$$

and $\alpha^{\perp} = \{b : (\exists c \in \alpha) b \wedge c \lhd_{\mathcal{M}(X)} \varnothing\}$ is the open complement of α in $\mathcal{M}(X)$.

Proposition

If $\alpha \in \mathrm{Pt}(\mathcal{M}(X)_{LS})$ *then* $\mathrm{Pt}([\alpha]) = \{x \in \mathrm{Pt}(\mathcal{M}(X)) : \alpha \subseteq x\}$

Locally compact metric spaces

In case $\alpha = {\Uparrow} b(x,\delta)$ the Proposition implies

$$\mathrm{Pt}([\alpha]) = B[x,\delta].$$

The same is true for the more general closed ball points

$$b[\alpha, r] = \{ b(x,\delta) \in M : b(x,\delta) \in \alpha \ \& \ m(j(x),\alpha) + \beta < \delta \}$$

where $\alpha \in \mathrm{Pt}(\mathcal{M}(X))$ and $r \in \mathbb{R}_{\geq 0}$,

An arbitrary $\alpha \in \mathrm{Pt}(\mathcal{M}(X)_{LS})$ is then a directed intersection of closed balls:

$$\mathrm{Pt}([\alpha]) = \bigcap_{b(x,\delta)\in\alpha} B[x,\delta]$$

Continuous real valued maps

Theorem (Coquand & Palmgren & Spitters, 2009)

If a closed subspace $\mathcal{S} = \mathcal{M}(X) \smallsetminus U$ is overt and $u \in Pos_{\mathcal{S}}$, then there is $\alpha \in \mathrm{Pt}(\mathcal{S}) \cap u^*$. Moreover, if $\mathcal{S}$ is also compact then $\mathrm{Pt}(\mathcal{S})$ is complete and totally bounded.

Lemma

If X is locally compact and $\gamma = b[\alpha, r] \in \mathrm{Pt}(\mathcal{M}(X)_{LS})$ for $\alpha \in \mathcal{M}(X)$ and $r \in \mathbb{R}_{\geq 0}$. Then $[\gamma]$ defines a compact overt subspace and hence $A = \mathrm{Pt}([\gamma])$ is complete and totally bounded.

If $F : \mathcal{M}(X) \to \mathcal{R}$ the quantities $\inf_{x \in A} \mathrm{Pt}(F)(x)$ and $\sup_{x \in A} \mathrm{Pt}(F)(x)$ exist (E. Bishop & D. Bridges, 1985).

Continuous real valued maps

For $F : \mathcal{M}(X) \to \mathcal{R}$ we denote $f = \mathrm{Pt}(F)$ and $\widehat{f} = \mathrm{Pt}(F_{LS})$.

Proposition

Let X be a locally compact normed vector space and $F : \mathcal{M}(X) \to \mathcal{R}$ a continuous relation. Let $\gamma = b[\alpha, r] \in \mathrm{Pt}(\mathcal{M}(X)_{LS})$ and let $A = \mathrm{Pt}([\gamma]) \subseteq \mathrm{Pt}(\mathcal{M}(X))$ then

$$\widehat{f}(b(\alpha, r)) = \left[\inf_{x \in A} f(x), \sup_{x \in A} f(x)\right]$$

This determines the function $\widehat{f}$, since the collection of points $\Uparrow b(x, \delta)$ is a base for the domain $(\mathrm{Pt}(\mathcal{M}(X)_{LS}), \subseteq)$.

Partial reals

The formal reals $\mathcal{R}$ is the localic completion $\mathcal{M}(\mathbb{Q})$, wrt to the usual metric on $\mathbb{Q}$. Equivalently basic opens are pairs $(p,q) \in \mathbb{Q} \times \mathbb{Q}$, such that $p < q$ (corresponding to ball symbols $b(\frac{p+q}{2}, \frac{q-p}{2})$). The pre-order and strict inclusion are

$$(p,q) \leqslant (r,s) \Leftrightarrow r \leq p < q \leq s$$

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Taking $\mathcal{R}_{LS}$ we get precisely the partial reals $\mathcal{R}_P$ (Negri, 2002). Its points correspond to the generalized reals in (Richman, 1998). These are cuts (L,U) of disjoint open subsets of $\mathbb{Q}$ with L downwards and U upwards closed wrt $< \subseteq \mathbb{Q} \times \mathbb{Q}$.

A partial real α is the same as a (bounded) cut

$$(\{p : (\exists q \in \mathbb{Q})(p,q) \in \alpha\}, \{q : (\exists p \in \mathbb{Q})(p,q) \in \alpha\})$$

Partial reals and interval analysis

The closed ball points $b[\alpha, r]$ are now closed intervals $[\alpha - r, \alpha + r]$.

As a corollary to the previous Proposition the arithmetic operations on $\mathbb{R}$ lift to the ordinary interval arithmetic operations, so e.g.

$$[x, y] + [z, w] = [x + z, y + w]$$

and

$$[x, y] \cdot [z, w] = [\min(xz, xw, yz, yw), \max(xz, xw, yz, yw)]$$

Proposition

If $\alpha \in \mathrm{Pt}(\mathcal{R}_P)$ then $\alpha = [x, y]$ for some reals $x \leq y$ if and only if $[\alpha]$ is overt.

Partial reals and interval analysis

Classically the partial reals are closed intervals ordered under reverse inclusion, i.e. the interval domain $\mathbf{I}\mathbb{R}$, without a bottom element $\bot = \mathbb{R}$.

We can add a bottom to $\mathrm{Pt}(\mathcal{R}_{LS})$ as follows: In $\mathcal{R}$ we take basic pairs in $(\mathbb{Q} \cup \{-\infty\}) \times (\mathbb{Q} \cup \{+\infty\})$ and let

$$(p, q) \prec (r, s) \Leftrightarrow r < p < q < s \text{ or } (r, s) = (-\infty, +\infty).$$

Then $\bot = \{(-\infty, +\infty)\} \in \mathrm{Pt}(\mathcal{R}_{LS})$ is the least element.

Using (classical) domain theory a lot of work has been done on effective interval analysis (Edalat et al). A suggestion is then to use the formal space $\mathrm{Pt}(\mathcal{R}_{LS})$ and the above lifting result to develop (parts of) this theory constructively.

Partial reals and interval analysis

In (Edalat & Lieutier, 2004) the authors define a derivative for Scott continuous functions $f : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$.

Let $a, b \in \mathbf{I}\mathbb{R}$. A function $f : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$ has an interval Lipschitz constant b in a, if for all $x_1, x_2 \gg a$

$$b(x_1 - x_2) \sqsubseteq f(x_1) - f(x_2).$$

Denote this relation by $f \in \delta(a, b)$.

For $f : \mathbb{R} \to \mathbb{R}$, we have $\mathbf{I}f \in \delta([a_1, a_2], [b_1, b_2])$ iff f is lipschitz on (a_1, a_2) and for all $u, v \in (a_1, a_2)$ with $v \leq u$ we have

$$b_1(u - v) \leq f(u) - f(v) \leq b_2(u - v).$$

Partial reals and interval analysis

For $a, b \in \mathbf{I}\mathbb{R}$ the single step function $a \searrow b$ is defined by

$$(a \searrow b)(x) = \begin{cases} b, & \text{if } a \ll x \\ \bot, & \text{otherwise} \end{cases}$$

The domain derivative of a function $f : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$ is then defined as

$$\frac{df}{dx} = \bigsqcup_{f \in \delta(a,b)} a \searrow b$$

Theorem (Edalat & Lieutier, 2004)

- $\frac{df}{dx}$ *is well-defined and Scott continuous for each* f
- *If* $f \in C^1$ *then* $\frac{d\mathbf{I}f}{dx} = \mathbf{I}(f')$
- $f \in \delta(a, b)$ *iff* $a \searrow b \sqsubseteq \frac{df}{dx}$

Partial reals and interval analysis

The situation for continuous functions $f : \mathrm{Pt}(\mathcal{R}_P) \to \mathrm{Pt}(\mathcal{R}_P)$ is similar.

To show that $\frac{df}{dx}$ is well defined for every $f : \mathrm{Pt}(\mathcal{R}_P) \to \mathrm{Pt}(\mathcal{R}_P)$, i.e. that the set $\{a \searrow b : f \in \delta(a, b)\}$ is consistent, we can describe the supremum $\frac{df}{dx}$ as the point function of a continuous relation $\mathcal{R}_P \to \mathcal{R}_P$.

Given a continuous relation $F : \mathcal{R}_\mathbf{P} \to \mathcal{R}_\mathbf{P}$ we define a new relation $d_F \subseteq R \times R$ by

$$\begin{aligned}(p,q) d_F (r,s) \quad \overset{\text{def}}{\Leftrightarrow} \quad & [\forall (x_1, y_1), (x_2, y_2) \prec (p,q)]\,[\exists (u_1, v_1), (u_2, v_2)] \\ & (x_1, y_1) F (u_1, v_1) \ \&\ (x_2, y_2) F (u_2, v_2) \ \& \\ & (u_1, v_1) - (u_2, v_2) \prec (r,s)((x_1, y_1) - (x_2, y_2)).\end{aligned}$$

Partial reals and interval analysis

Then we define $D_F \subseteq R \times R$ by setting

$$(p,q) D_F (r,s) \overset{\text{def}}{\Leftrightarrow} \{(p,q)\}_{\prec} \subseteq I_F^{-1}\{(r,s)\}_{\prec}.$$

Proposition

The relation D_F is a continuous relation $\mathcal{R}_{\mathbf{P}} \to \mathcal{R}_{\mathbf{P}}$ whenever F is. Moreover, we have $(p,q)D_F(r,s)$ if and only if for every $(p',q') \prec (p,q)$ there is $(r',s') \prec (r,s)$ such that $Pt(F) \in \delta(I[p',q'], I[r',s'])$.

Corollary

Let $F : \mathcal{R}_{\mathbf{P}} \to \mathcal{R}_{\mathbf{P}}$ be a continuous relation. Then we have

$$Pt(D_F) = \frac{df}{dx}$$

where $f = \mathrm{Pt}(F)$.

Further work

The operator $\frac{d}{dx}$ is used to give a domain theoretic generalization of the Picard-Lindelöf Theorem:

The Picard-Lindelöf Theorem guarantees a unique solution to

$$(*) \quad \begin{cases} f'(t) = v(t, f(t)) \\ f(t_0) = x_0 \end{cases}$$

if v is Lipschitz in the second argument.
The solution is the unique fix-point of $P : f \mapsto \lambda t.(x_0 + \int_{t_0}^{t} v(s, f(s))ds)$.
If we let

$$A_v(f, g) = (f, \lambda t.v(t, f(t))), \quad U(f, g) = (\lambda t.(x_0 + \int_{t_0}^{t} g(s)ds), g)$$

we have $P(f) = \pi_0(U \circ A_v(f, g))$ for any g. The unique fix-point (f, g) of $U \circ A_v$ then satisfies $f' = g = \lambda t.v(t, f(t))$.

Further work

The domain theoretic Picard operator (corresponding to $U \circ A_v$) allows imprecise initial condition or function v:
Let $f_0, g_0 : \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$ and $v : \mathbf{I}\mathbb{R} \times \mathbf{I}\mathbb{R} \to \mathbf{I}\mathbb{R}$, Scott continuous with $g_0 \sqsubseteq \lambda t.v(t, f_0(t))$ (f_0 and g_0 also consistent).
The Picard operator then has a (least) fixed point $(f_s, g_s) \sqsupseteq (f_0, g_0)$. Here v need only be continuous.

If f is a solution to the classical problem $(*)$, then the domain theoretic solution f_s, computed using the canonical extension of v, satisfies $f(t) \in f_s(t)$ for all t in a neighborhood of t_0.

If v is computable and continuous but not Lipschitz, $(*)$ might have no computable solution (O. Aberth, 1971). However, the interval valued solution f_s exists and is computable.

Further work

Interesting to investigate the same question in the context of partial reals.

- Starting from imprecise initial data can we get a partial real valued solution to $(*)$ (similar to the domain theoretic solution)?
- Under what conditions on the initial value and function v can we estimate the width of the solution? When does it take interval values?
- Relation to solutions of the classical problem $(*)$?

Some references

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