# An Intuitionistic Epistemic Logic

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# Motivation

Applying program extraction (modified realizability) to generating asynchronously communicating programs.

# A standard reference: *Reasoning about Knowledge* [Fagin et al., 2003]

Warning: for the speaker, the formalisation below is complicated.

Let us fix
$\Phi$: a set of propositional variables.
$L_i$: a set (of local states) for $1 \leqslant i \leqslant n$.

$\mathcal{G} = L_1 \times \cdots \times L_n$ (global states).
A run over $\mathcal{G}$ is a function $\mathbb{N} \to \mathcal{G}$.
A system $\mathcal{R}$ over $\mathcal{G}$ is a set of runs $\mathcal{R} \subseteq \mathcal{G}^{\mathbb{N}}$.

An interpreted system $\mathcal{I}$ is a pair $(\mathcal{R}, \pi)$

- $\mathcal{R}$: a system over $\mathcal{G}$.
- $\pi : \mathcal{G} \to \Phi \to \{\top, \bot\}$.

# An interpreted system interprets the formulae

(still from [Fagin et al., 2003])

With the natural projection $f_i = \mathcal{G} \to L_i$,
$s \sim_i s'$ iff $f_i(s) = f_i(s')$.
A point: $(r, m) \in \mathcal{R} \times \mathbb{N}$.

- $(r, m) \models l$ iff $\pi(r, m)(l) = \top$ for $l \in \Phi$.
- $(r, m) \models \bot$ never holds.
- $(r, m) \models K_i\varphi$
  iff $(r', m') \models \varphi$ for any point $(r', m)$ such that
  $(r, m) \sim_i (r', m')$.
- $(r, m) \models \Box\varphi$ iff $(r, m') \models \varphi$ for all $m' \geqslant m$.
- $(r, m) \models \Diamond\varphi$ iff $(r, m') \models \varphi$ for some $m' \geqslant m$.
- $(r, m) \models K_i\varphi$
  iff $(\mathcal{I}, r', m') \models \varphi$ for any point $(r', m)$ such that
  $(r, m) \sim_i (r', m')$.
- $(r, m) \models \varphi \supset \psi$ iff $(r, m) \not\models \varphi$ or $(r, m) \models \psi$.

# Asynchronous communication in [Fagin et al., 2003]

A class $\mathcal{C}_n^{amp}$ of interpreted systems called asynchronous message-passing systems.

A history $h$ over $\Sigma_i, INT_i$ and $MSG$ is a nonempty finite sequence with

- $h_0 \in \Sigma_i$
- $h_k \in \{send(\mu, j, i), receive(\mu, j, i) \mid \mu \in MSG, 1 \leqslant j \leqslant n\}$ $\cup \{int(a, i) \mid a \in INT_i\}$ for $k > 0$.

Let $L_i (1 \leqslant i \leqslant n)$ be a prefix-closed set of histories.

Let $\mathcal{R}$ be the set of runs $r$ satisfying

- $f_i(r(0))$ is a history of length one.
- $f_i(r(m+1))$ is identical to $f_i(r(m))$ or a history obtained by appending one element to $f_i(r(m))$
- for every $receive(\mu, j, i)$ appearing in $f_i(r(m))$, there exists an event $send(\mu, i, j)$ appearing in $f_j(r(m))$.

$\mathcal{I} = (\mathcal{R}, \pi)$ is a.m.p. iff $\mathcal{R}$ can be constructed in this way.

### Axiomatisable?

“At this point, we do not even have a candidate for a sound and complete axiomatization of $\mathcal{C}_n^{amp}$”. [Fagin et al., 2003, Notes, Ch. 8]

# An important observation in [Fagin et al., 2003]

The processes can gain or lose knowledge only by sending and receiving messages.

This (ignoring “sending and”) seemed intuitionistic to the speaker.

# Extending Browuer–Heyting–Kolmogorov Interpretation with Communication

Browuer–Heyting–Kolmogorov interpretation taken from [Troelstra and van Dalen, 1988]

(H1) A proof of $\varphi \wedge \psi$ is given by presenting a proof of $\varphi$ and a proof of $\psi$.

(H2) A proof of $\varphi \vee \psi$ is given by presenting either a proof of $\varphi$ or a proof of $\psi$ (plus the stipulation that we want to regard the proof presented as evidence for $\varphi \vee \psi$ [plus left or right information]).

(H3) A proof of $\varphi \supset \psi$ is a construction which permits us to transform any proof of $\varphi$ into a proof of $\psi$.

(H4) Absurdity $\perp$ (contradiction) has no proof; a proof of $\neg\varphi$ is a construction which transforms any hypothetical proof of $\varphi$ into a proof of a contradiction.

# Extending Browuer–Heyting–Kolmogorov Interpretation with Communication

(HK) A proof of $K_p\varphi$ is a construction that witnesses agent $p$'s acknowledgement of a proof of $\varphi$ and also contains the acknowledged proof.

(H1) A proof of $\varphi \wedge \psi$ is given by presenting a proof of $\varphi$ and a proof of $\psi$.

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(H3) A proof of $\varphi \supset \psi$ is a construction which permits us to transform any proof of $\varphi$ into a proof of $\psi$.

(H4) Absurdity $\bot$ (contradiction) has no proof; a proof of $\neg\varphi$ is a construction which transforms any hypothetical proof of $\varphi$ into a proof of a contradiction.

# An Anonymous Refree's Comment

> this is at odds with real life applications, where an agent often has just disjunctive knowledge, So he can e.g. have a proof (evidence) that John works for either CIA or FBI, without having a prooof that he works for CIA or that he works for FBI.

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- The author should have explained BHK-interpretation in detail.
- FBI and CIA have secrets so that they sometimes make non-constructive proofs?
- Let us take a notion of proof satisfying BHK-interpretation.

# New informal reading of $K_p\varphi$

Formulae $\varphi ::= \bot \mid I \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \supset \varphi \mid K_p\varphi$.
all interpretable in classical epistemic logic
(widely attributed to [Hintikka, 1962], writes [Ditmarsch et al., 2007]).

$K_p\varphi$: $p$ knows $\varphi$. (What does "know" mean?)

**Classical** In all $p$'s possible worlds, $\varphi$ is true.

**This work** $p$ has received a proof of $\varphi$.

c.f. Plato: *Theaetetus*.

1. Knowledge is perception
2. Knowledge is a true opinion
3. Knowledge is a true opinion with explanation

# New informal reading of $K_q K_p \varphi$: COMMUNICATION

$K_q K_p \varphi$: $q$ knows that $p$ knows $\varphi$.

Classical In all $q$'s possible worlds, in all $p$'s possible worlds, $\varphi$ is true.
(Maybe useful for a philosopher defending conventionalism following David Lewis.)

This work $q$ has received a proof of the fact that $p$ has received a proof of $\varphi$.
Communication from $p$ to $q$

# Do we have this: $(K_p(\varphi \vee \psi)) \supset (K_p\varphi \vee K_p\psi)$

nnnn Analysis of the creative subject by [Dummett, 2000, p.237].

$$\forall n((\vdash_n A) \wedge (\vdash_n B) \rightarrow (\vdash_n (A \wedge B)))$$

is “less evident” than

$$\forall n((\vdash_n A) \vee (\vdash_n B) \leftrightarrow (\vdash_n (A \vee B))).$$

Analogously,

$$K_p(A \wedge B) \supset (K_p(A \wedge B))$$

is less evident than

$$K_p(A \vee B) \supset (K_p(A \vee B)).$$

# Deduction System

$$(\mathrm{T})\ \frac{\Gamma \vdash K_p\varphi}{\Gamma \vdash \varphi} \qquad (\mathrm{introspection})\ \frac{\Gamma \vdash K_p\varphi}{\Gamma \vdash K_pK_p\varphi}$$

$$(\mathrm{necessitation})\ \frac{\Gamma \vdash \varphi}{K_p\Gamma \vdash K_p\varphi} \qquad (\vee K)\ \frac{\Gamma \vdash K_p(\varphi \vee \psi)}{\Gamma \vdash K_p\varphi \vee K_p\psi}$$

$$(\mathrm{ax})\ \frac{}{\varphi \vdash \varphi} \qquad (\mathrm{w})\ \frac{\Gamma \vdash \varphi}{\psi,\ \Gamma \vdash \varphi} \qquad (\mathrm{c})\ \frac{\varphi,\ \varphi,\ \Gamma \vdash \varphi'}{\varphi,\ \Gamma \vdash \varphi'} \qquad (\mathrm{e})\ \frac{\Gamma,\ \varphi,\ \psi,\ \Gamma' \vdash \varphi'}{\Gamma,\ \psi,\ \varphi,\ \Gamma' \vdash \varphi'}$$

$$(\wedge\text{-E}_0)\ \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi} \qquad (\wedge\text{-I})\ \frac{\Gamma \vdash \varphi \qquad \Gamma' \vdash \psi}{\Gamma, \Gamma' \vdash \varphi \wedge \psi} \qquad (\wedge\text{-E}_1)\ \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}$$

$$(\vee\text{-I}_0)\ \frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \qquad (\vee\text{-I}_1)\ \frac{\Gamma \vdash \varphi}{\Gamma \vdash \psi \vee \varphi}$$

$$(\vee\text{-E})\ \frac{\Gamma \vdash \psi_0 \vee \psi_1 \qquad \Gamma,\ \psi_0 \vdash \varphi \qquad \Gamma,\ \psi_1 \vdash \varphi}{\Gamma \vdash \varphi}$$

$$(\supset\text{-I})\ \frac{\varphi,\ \Gamma \vdash \psi}{\Gamma \vdash \varphi \supset \psi} \qquad (\supset\text{-E})\ \frac{\Gamma \vdash \psi_0 \supset \psi_1 \qquad \Gamma \vdash \psi_0}{\Gamma \vdash \psi_1} \qquad (\bot\text{-E})\ \frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}$$

If we add the double negation elimination, we obtain $\varphi \supset K_p\varphi$.

# Formal Semantics = Intuitionistic Logic (with Knowledge)

model $\langle W, \preceq, (f_p)_{p \in P} \rangle$
$f_p \colon W \to W$: idempotent, decreasing, monotonic

valuation $\rho \colon \mathrm{PVar} \to \mathcal{P}(W)$ $\rho(I)$: upward-closed

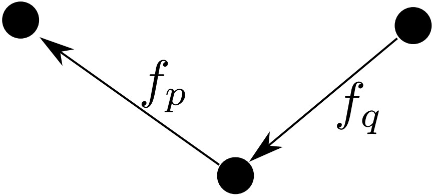
Define $w \models \varphi$ for a state $w \in W$ and a formula $\varphi$:

$$\begin{aligned}
w \models \bot \quad &\Leftrightarrow \quad \text{never} \\
w \models I \quad &\Leftrightarrow \quad w \in \rho(I) \\
w \models K_p\psi \quad &\Leftrightarrow \quad f_p(w) \models \psi \\
w \models \psi_0 \wedge \psi_1 \quad &\Leftrightarrow \quad \text{both } w \models \psi_0 \text{ and } w \models \psi_1 \text{ hold} \\
w \models \psi_0 \vee \psi_1 \quad &\Leftrightarrow \quad \text{either } w \models \psi_0 \text{ or } w \models \psi_1 \text{ holds} \\
w \models \psi_0 \supset \psi_1 \quad &\Leftrightarrow \quad v \models \psi_0 \text{ implies } v \models \psi_1 \text{ for any } v \succeq w.
\end{aligned}$$

# Formal Semantics = Intuitionistic Logic (+ Knowledge)

model $\langle W, \preceq, (f_p)_{p \in P} \rangle$
$f_p \colon W \to W$: idempotent, decreasing and monotonic

past $\preceq$ future



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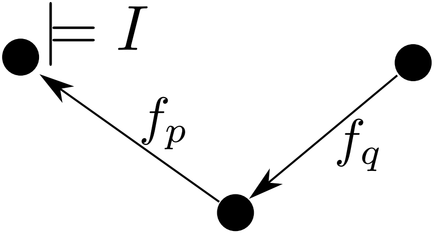

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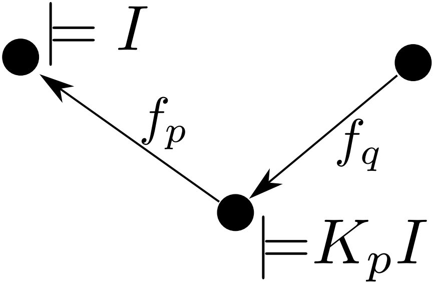

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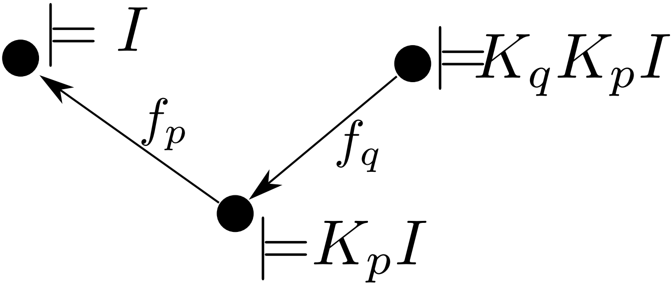

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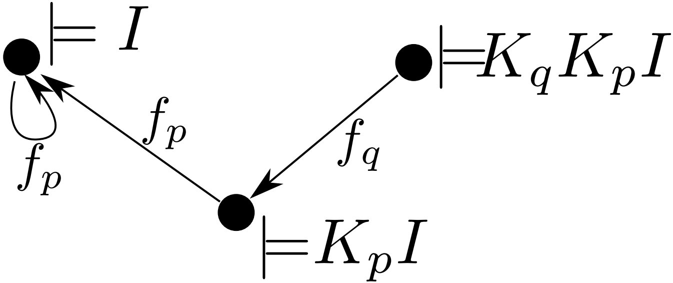
$p$'s state.



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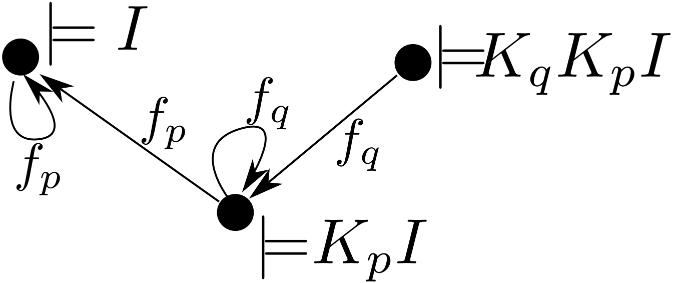
past $\preceq$ future

$p$'s state. $q$'s state.



# Soundness and Strong Completeness

$$\Gamma \models \varphi \quad \Longleftrightarrow \quad \Gamma \vdash \varphi.$$

## Proof strategy

Following [Troelstra and van Dalen, 1988].

For a formula $\Gamma \not\vdash \varphi$,
we construct a model $M$ and a state $w \in M$
so that $M, w \models \Gamma$ but not $M, w \models \varphi$.

By
$W$ is the set of saturated sets of formulae.
$f_p(\Gamma) = \{\varphi \mid K_p\varphi \in \Gamma\}$.
Checking $f_p$ is actually $W \to W$ requires the rule $(\vee K)$.

# Disjunction Property

$\vdash \varphi \vee \psi \quad \Longrightarrow \quad \vdash \varphi$ or $\vdash \psi$

## Proof strategy

By extending Aczel's slash relation [Troelstra and van Dalen, 1988] by

$$\Gamma \mid K_p\varphi \Longleftrightarrow f_p(\Gamma) \mid \varphi$$

where $f_p(\Gamma)$ (agent $p$'s view on a set of formulae $\Gamma$) defined as

$$\begin{aligned} g_p(\Gamma) &= \{\varphi \in \mathrm{Fml} \mid (K_p)^+\varphi \in \Gamma \text{ and } \varphi \text{ does not begin with } K_p\}, \\ f_p(\Gamma) &= g_p(\Gamma) \cup K_p g_p(\Gamma) \cup \{\varphi \in \mathrm{Fml} \mid \Gamma \vdash \bot\}. \end{aligned}$$

# Finite model property

$M \models \varphi$ for all finite $M \iff \vdash \varphi$.

## Proof strategy

For a formula $\nvdash \varphi$,
we construct a finite model $M$ and a state $w \in M$
so that $M, w \not\models \varphi$.

### It does not work:

only looking at the formulae in a subformula-closed set $\Omega$ and using one of the previous $f_p$'s.
Reason: $f_p \colon W \to W$ does not hold.

## Instead

$W$ to be the set of pairs $(\Omega, \Gamma)$ where $\Gamma$ is $\Omega$-saturated.
($\Omega$ is closed for taking a subformula and replacing $K_pK_p$ with $K_p$).

$F_p((\Omega, \Gamma)) = (f'_p(\Omega), f_p(\Gamma))$ where

- $f_p(\Gamma) = g_p(\Gamma) \cup K_p g_p(\Gamma) \cup \{\varphi \in \mathsf{Fml} \mid \Gamma \vdash \bot\}$.
- $f'_p(\Omega) = g_p(\Omega) \cup K_p g_p(\Omega)$.

meanwhile

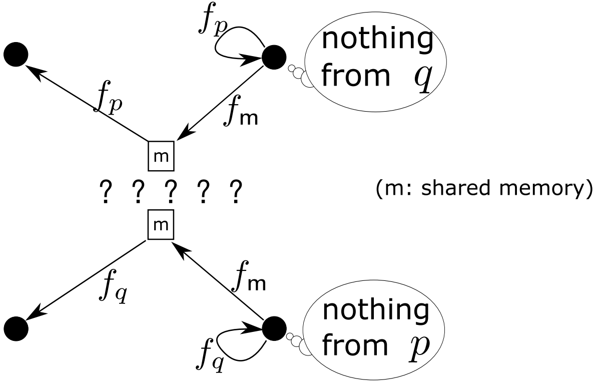
# Modelling Sequential Consistency

# Need for shared memory consistency

Assumption: full-information

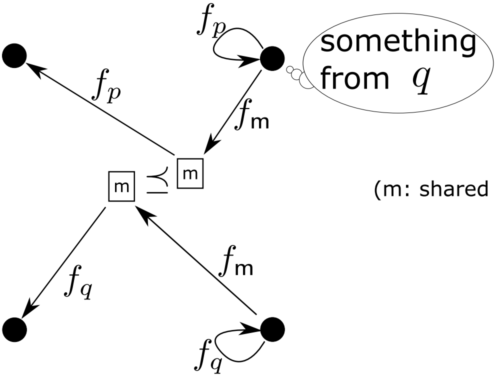
- A message contains all knowledge of its sender.
- Nothing is ever forgotten.

Even under this assumption, no communication is guaranteed between processes.



# Essence of Sequential Consistency

For two memory states, either $\preceq$ or $\succeq$ holds.



(m: shared memory)

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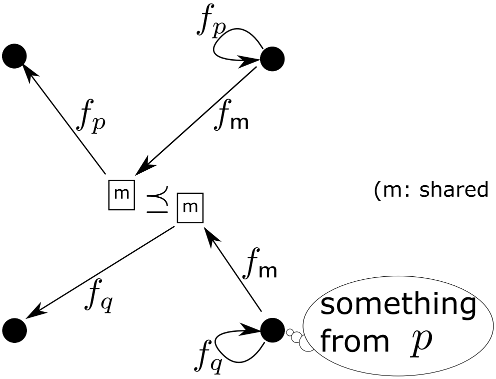

(m: shared memory)

# Logical Background: logic **Lin** for linear models

**Lin** $=$ Intuitionistic logic $+ (\varphi \supset \psi) \vee (\psi \supset \varphi)$:

$$\text{Intuitionistic logic} \subsetneq \textbf{Lin} \subsetneq \text{Classical logic}$$

Well-known property:

$$\textbf{Lin} \vdash \theta \iff M \models \theta \text{ for all linear model } M$$

(Linear model: for any two states, either $\preceq$ or $\succeq$ holds.)

# A logic **SC** for Sequential Consistency

**SC** = Int. Epistemic logic + $(K_{\mathrm{m}}\varphi \supset K_{\mathrm{m}}\psi) \vee (K_{\mathrm{m}}\psi \supset K_{\mathrm{m}}\varphi)$:

Intuitionistic epistemic logic $\subsetneq$ **SC** $\subsetneq$ Classical logic

A result:

$$\mathbf{SC} \vdash \theta \iff M \models \theta \text{ for all sequential model } M$$

(Sequential model: for any two memory states, $\preceq$ or $\succeq$ holds.)

# An example theorem under sequential consistency

$$\vdash ((K_p K_m K_p I) \wedge K_q K_m K_q J) \supset ((K_q K_p I) \vee K_p K_q J)$$

## Informal reading

- $p$ sends a proof of $I$ to $m$, then $m$ replies to $p$.
- $q$ sends a proof of $J$ to $m$, then $m$ replies to $q$.
- then, $p$'s knowledge $I$ has been transmitted to $q$, or $q$'s knowledge $J$ has been transmitted to $p$.

# Ongoing work: finite sequential model property of **SC**

Trying to avoid

- logically possible but computationally impossible schedules like

$$\overbrace{t_0 \lnsim t_1 \lnsim t_2 \lnsim \cdots \lnsim t_n \lnsim \cdots}^{\text{infinite}} \lnsim t'$$

- finite but non-sequential schedules.

Revising a proof until the speaker finds a gap.
(Similar construction using $f_p(\Gamma) = \{\varphi \mid \Gamma \nvdash (K_p\varphi) \supset \bot\}$).

If succeeds, a similar method would give an axiomatization for Halpern's $\mathcal{C}_n^{amp}$ limited to the class of formulae whose every subformula

- begins with $\Box$, or
- is immediately after $\Box$.

Def. **AMP** = Int. Epis. logic + $(K_p\varphi \supset K_p\psi) \vee (K_p\psi \supset K_p\varphi)$.
Speculation. **AMP** $\vdash \varphi \iff \mathcal{C}_n^{amp} \models (\varphi)^\circ$
where $(\varphi)^\circ$ is obtained by putting $\Box$ before every subformula of $\varphi$.

# Future Work

Extending program extraction to concurrent/distributed computation.

- Making proofs constructive.
- Modelling other memory consistencies: especially PRAM consistency, cache consistency and processor consistency
- Typed lambda calculus
  - Type-safe Paxos [Lamport, 1997] implimentation
- Quantify agents $\exists x K_x \varphi$ for program extraction with mobility.
  - Knowledge of $\pi$-calculus terms
- Knowledge of forking and merging agents (forking creates common knowledge).

A part (soundness, strong completeness and modelling sequential consistency) of this work has been accepted to LPAR-16 that will be held in Dakar, Senegal.

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