some existence axioms in finite order arithmetic

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Contents

Finite order arithmetic is a formal system based on λ-culiculus.

sorts

1. $\mathcal{M}_0 \quad \longleftrightarrow \mathbb{N}$
2. $\mathcal{M}_{\sigma\to\tau} \longleftrightarrow$ the set of all maps $\mathcal{M}_\sigma$ to $\mathcal{M}_\tau$

where σ and τ are given sorts.

In short, $0 \to 0$ is denoted by 1. similarly $n \to 0$ is denoted by $n+1$. $\sigma_1 \to (\sigma_2 \to \tau)$ is denoted by $(\sigma_1, \sigma_2) \to \tau$.

terms

- $\lambda x^{\sigma}.t^{\tau}$ (the sort is $\sigma \to \tau$.)
- $t^{\sigma\to\tau}(s^{\sigma})$ (the sort is τ.)

where t and s are given terms, x is a variable symbol.

axiom of λ-caliculus

- (λ-reduction)

$$(\lambda x^{\sigma}.t^{\tau})(s^{\sigma}) = t[s/x].$$

- (extentionality)

$$\forall x^{\sigma\to\tau}\forall y^{\sigma\to\tau}(x = y \leftrightarrow \forall z^{\sigma}(x(z) = y(z))).$$

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translation from S.O.A. to F.O.A.

$\mathcal{M}$: A model of finite order arithmetic.

$\longrightarrow (\mathcal{M}_0, \{X \in \mathcal{M}_1 | \forall n \in \mathcal{M}_0 (X(n) \in \{0, 1\})\}).$

translation from F.O.A. to set theory

V: A model of set theory.

$$\longrightarrow \begin{cases} \mathcal{M}_0 &= \omega^V, \\ \mathcal{M}_{\sigma\to\tau} &= \{f : \mathcal{M}_\sigma \to \mathcal{M}_\tau\}^V. \end{cases}$$

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Strength of S.O.A., F.O.A. and set theory is as follows.

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- (Kohlenbach, 2005) An axiom RCA_0^ω of F.O.A., our base axiom, is conservative extension of an axiom RCA_0 of S.O.A.
- (Hunter, 2008) $\mathrm{RCA}_0^\omega + (\mathcal{E}_1)$ is conservative extension of ACA_0.
- (Hunter, 2008) $\mathrm{RCA}_0^\omega + (\mathcal{E}_2)$ is conservative extension of Z_2.

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- $\mathrm{ZF} \vdash \mathrm{RCA}_0^\omega + (\mathcal{E}) + \mathrm{Con}(\mathrm{RCA}_0^\omega + (\mathcal{E}))$.
- $\mathrm{ZFC} \vdash \mathrm{RCA}_0^\omega + (\mathcal{E}) + \mathrm{AC} + \mathrm{Con}(\mathrm{RCA}_0^\omega + (\mathcal{E}) + \mathrm{AC})$.

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Advantage of finite order arithmetic

- Abstract mathematics can be considerable. (If we do not fix the sort, the mood of arbitrary set could be expressed.)
- Many axioms (e.g. axiom of comprehension, choice, recursion or continuum hypothesis) are different for each sort. Finer analysis than set theory could be done.

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2. Definitions of axiom of finite order arithmetic.

1. Introduction.
   - What is finite order arithmetic
   - F.O.A is between S.O.A. and Set Theory
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   - The base axiom RCA_0^ω
   - the axiom of comprehension and choice
3. Higher rank axioms imply lower rank axioms.
4. The hierarchy of comprehension does not collapse.

Definition. RCA_0^ω is the axiom consists of the following formulas.

- The axiom of λ-calculus.
- $\forall x^0(\exists y^0(x = S(y)) \leftrightarrow x \neq 0), \forall x^0 \forall y^0(S(x) = S(y) \to x = y)$
- (Existence of primitive recursion operator.)

$$\exists \mathcal{R}_0 \forall f^1 \forall n^0 \forall m^0 \left[\begin{array}{rcl} \mathcal{R}_0(f,n)(0) & = & n, \\ \mathcal{R}_0(f,n)(S(m)) & = & f(m, \mathcal{R}_0(f,n)(m)). \end{array} \right]$$

- (Induction axiom.)

$$\forall A^1(0 \in A \wedge \forall n^0(n \in A \to S(n) \in A) \to \forall n(n \in A)).$$

- (Axiom of choice for $(1,0)$.)

$$\forall A^{(1,0)\to 0}[(\forall x^1 \exists y^0 (x,y) \in A) \to (\exists F^{1\to 0} \forall x (x, F(x)) \in A)].$$

Where 0^0 and S^1 are constant symbols.

Definition

- (Q$^\sigma$-comprehension): $\exists X^{\tau\to 0}\forall x^\tau(x \in X \leftrightarrow \varphi(x))$
  where φ is described by $=_0$, Boolean connections and σ variable quantifier $\exists y^\sigma, \forall y^\sigma$.
- $(\mathrm{AC}^{\sigma,\tau})$: $\forall A^{(\sigma,\tau)\to 0}(\forall x^\sigma \exists y^\tau((x,y) \in A) \to \exists F^{\sigma\to\tau}((x,F(x)) \in A))$.
- $(\mathcal{E}_\sigma) : \exists E^{\sigma\to 0}\forall x^\sigma(x \in E \leftrightarrow \forall y^{\sigma'} x(y) = 0)$
  where $\sigma = \sigma' \to 0$.

Proposition.

Q$^\sigma$-comprehension is equivalent to $(\mathcal{E}_{\sigma\to 0})$ under RCA_0^ω.

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Definition (the rank of sort)

The rank of sort is defined as follows inductively.

$$\begin{aligned} rank(0) &:= 0 \\ rank(\sigma \to \tau) &:= \max(rank(\sigma) + 1, \mathrm{rank}(\tau)) \end{aligned}$$

Intuitively, rank is corresponded to the cardinality of the set of all elements. $rank(0) = 0$ means $\mathcal{M}_0$ is countable, rank $= 1$ means continuum, rank $= 2$ is to have cardinality of power set of continuum...

Lemma.

Assume $rank(\sigma) \leq rank(\sigma')$, then the assertion

$$\exists I^{\sigma \to \sigma'} \exists P^{\sigma' \to \sigma} \forall x^{\sigma} (P(I(x)) = x)$$

is provable in RCA_0^{ω}.

Proposition.

Let $\sigma, \sigma', \tau, \tau'$ be sorts and assume $rank(\sigma) \leq rank(\sigma'), rank(\tau) \leq rank(\tau')$. Then the following statements are provable in RCA_0^{ω}.

1. $(\mathcal{E}_{\sigma'}) \to (\mathcal{E}_{\sigma})$.
2. $\mathrm{AC}^{\sigma', \tau'} \to \mathrm{AC}^{\sigma, \tau}$.

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Theorem.

Let $n \geq 1, 0 \leq k \leq n-2$ and $0 \leq l \leq n-1$. Then the following holds.

$$\mathrm{RCA}_0^\omega + (\mathcal{E}_{n+1}) + \mathrm{AC}^{k,l} \vdash \mathrm{Con}(\mathrm{RCA}_0^\omega + (\mathcal{E}_n) + \mathrm{AC}^{k,l}).$$

Thus, $\mathrm{RCA}_0^\omega + (\mathcal{E}_n)$ does not imply $(\mathcal{E}_{n+1})$.

The idea of proof: Fix a model $\mathcal{M}$ of $\mathrm{RCA}_0^\omega + (\mathcal{E}_{n+1}) + \mathrm{AC}^{k,l}$. The theorem is proved by some construction in $\mathcal{M}$. It is consists of 3 steps.

1. To construct a model consists of all ”terms” described in some constants.
2. To construct the interpretation of lower sorts elements.
3. To construct the graph of the truth value function and to check the axioms.

(continue)

First step: the construction of the term model.
We construct the model in $\mathcal{M}$; Let $\mathcal{N}$ be the minimum set satisfies

- $\bigcup_{j \leq n-1} \mathcal{M}_j \cup \{S, \mathcal{R}_0, E_n\} \cup \{\text{variable symbols}\} \subset \mathcal{N}$
- closed under λ-introduction and application

The element of $\mathcal{N}$ can be coded in $\mathcal{M}_{n-1}$ since the existence of embeddings $\mathcal{M}_j \to \mathcal{M}_{n-1}$. $t \in \mathcal{N}$ is described as ”the construction of t is compatible to term construction conditions”. So $t \in \mathcal{N}$ can be defined by $(\mathcal{E}_n)$(= comprehension for rank $n-1$ quantifier formulas).

(continue)

Second step: the construction of the interpretation of lower sorts. We construct "the interpretation" $\mathcal{N}$ in $\mathcal{M}$.
Intuitively, define the graph of interpretation maps $h^\tau : \mathcal{N}_\tau \to \mathcal{M}_\tau$ $(\tau = 0, \cdots n-1, 0 \to (0 \to 0))$. $\{h^\tau\}_\tau$ satisfy suitable conditions, for example,

- If t is the code for $a \in \mathcal{M}$ then $h(t) = a$.
- If $t = \lambda x.s$ then $h(t) = \lambda a.h(s[a'/x])$ where a' is the code for $a \in \mathcal{M}$.
- If $t = E_n(\lambda x^n.s^0)$ and $\mathcal{M} \models \exists a^n(h(s[a'/x]) \neq 0)$ then $h(t) = 1$.

In the real proof, h is defined by "primitive recursion";

$$\begin{aligned} h_0 &= \{(t,a) | t \text{ is the code for } a\}, \\ h_{j+1} &= h_j \cup \{(\text{described in } h_j \text{ and "the suitable conditions"})\}. \end{aligned}$$

So $(t, a) \in h$ can be defined by "there exists the construction of h less than depth of t steps". Therefore h is exists by $(\mathcal{E}_{n+1})$.

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Third step: construction of the graph of the truth value function.
We define the truth value of $\mathcal{N} \models \varphi$.

- $\mathcal{N} \models t^0 = s^0 \quad :\Longleftrightarrow h(t) = h(s)$.
- $\mathcal{N} \models \neg\varphi \quad :\Longleftrightarrow \mathcal{N} \not\models \varphi$.
- $\mathcal{N} \models (\varphi \wedge \psi) \quad :\Longleftrightarrow (\mathcal{N} \models \varphi)$ and $(\mathcal{N} \models \psi)$.
- $\mathcal{N} \models \forall x^\sigma \varphi(x) :\Longleftrightarrow \forall t \in \hat{\mathcal{N}}_\sigma(\mathcal{N} \models \varphi(t/x))$.

Where $\hat{\mathcal{N}}_\sigma$ is the set consists of all element of $\mathcal{N}$ such that the sort is σ and has no free variables. The existence of the truth value function is proved by $(\mathcal{E}_n)$.

$\{\varphi | \mathcal{N} \models \varphi\}$ is exists in $\mathcal{M}$, it is complete theory and includes $\mathrm{RCA}_0^\omega + (\mathcal{E}_{n+1}) + \mathrm{AC}^{k,l}$. □

Thank you for your attention!