

# Choquet type theorems and continuous domains

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# General framework

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Adapt methods of measure theory and functional analysis

- to continuous domains in the sense of D. S. Scott
- to a non Hausdorff setting

Slogan:

asymmetric topology  $\rightarrow$  asymmetric functional analysis

# Why?

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Domains as introduced by D.S. Scott for semantics of programming languages can be viewed alternatively as order or as topological structures. This point of view has been extended to other topologies: Stably compact spaces (A. Jung et al.), qcb-spaces (A. Simpson et al.)

All these spaces are far from being Hausdorff: They subsume order through the specialisation order ( $x \leq y$  iff  $x \in cl(\{y\})$ ) which is to be viewed as an order of increasing information (which introduces 'asymmetry').

A semantics for systems comprising nondeterministic and probabilistic features requires the development of power-domain constructions (hyperspaces, probabilistic powerdomains) for these spaces and to prove that they have the desired properties.

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# Some Basic Ideas

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Continuous domains  $\cong$

an order theoretical abstraction of a situation where all objects can be approximated from below by their relatively compact (finitary) parts.

Scott-continuous functions  $\cong$

functions preserving approximation from below.

Effectivisation through an enumeration of a countable 'basis' which allows to approximate 'computable' objects by a recursively enumerable set of their relatively compact parts belonging to the basis.

Methods of topology and analysis based mainly on compactness arguments are likely to carry over to the non-Hausdorff situation, but not methods being based on completeness arguments.

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# A Case Study: Distribution functions

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**Fact:** For a (positive bounded Borel) measure  $\mu$  on  $\mathbb{R}$  its *distribution function*  $F: \mathbb{R} \rightarrow \mathbb{R}_+$  defined by:

$$F(x) = \mu(] - \infty, x[)$$

has the following properties:

- (i)  $F$  is strict;  $\inf_{x \in \mathbb{R}} F(x) = 0$
- (ii)  $F$  is monotone:  $x \leq y \Rightarrow F(x) \leq F(y)$
- (iii)  $F$  is lower semicontinuous:  $x_i \nearrow x \Rightarrow F(x_i) \nearrow F(x)$

And every function with (i), (ii), (iii) is the distribution of a unique measure on  $\mathbb{R}$ .

Usefulness: Riemann Stieltjes integral

$$\int f d\mu = \int f dF = \lim_{\Delta x \rightarrow 0} \sum_i f(x_i) (F(x_i + \Delta x) - F(x_i))$$

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# Choquet's Theorem 1954

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$X$  a locally compact Hausdorff space,  
 $\mathcal{O}(X)$  the lattice of all open subsets  $U \subseteq X$ ,  
 $\mathcal{K}(X)$  the space of all nonempty compact subsets  $K \subseteq X$   
with the Vietoris topology generated by

$$\square U = \{K \mid K \subseteq U\} \text{ and } \diamond U = \{K \mid K \cap U \neq \emptyset\}$$

For a measure  $\mu$  on the hyperspace  $\mathcal{K}(X)$  its distribution function  $F: \mathcal{O}(X) \rightarrow \mathbb{R}$  defined by  $F(U) = \mu(\square U)$  has the following properties:

- (i)  $F$  is strict:  $F(\emptyset) = 0$
- (ii)  $F$  totally monotone: .....
- (iii)  $F$  lower semicontinuous:  $U_i \nearrow U \Rightarrow F(U_i) \nearrow F(U)$

And every such function is the distribution of a uniquely determined measure on  $\mathcal{K}(X)$ .

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# Problem

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For which spaces  $L$  can we characterize measures by their distribution functions?

There is a long paper by A. Revuz in the Annales de l'Institut Fourier 1956 dealing with this problem. The spaces that Revuz is coming up with look very much like continuous lattices.

**Claim** The setting of continuous lattices is appropriate to deal with the above problem.



# Dcpo

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= **D**irected **c**omplete **p**artially **o**rded set

= partially ordered set  $X$  in which every directed family  $x_i$  has a least upper bound  $x = \sup_i x_i$ ; write  $x_i \nearrow x$ .

A map  $g: X \rightarrow Y$  of dcpos is *Scott-continuous* if it is

- (i) monotone:  $x \leq y \Rightarrow f(x) \leq f(y)$
- (ii) lower semicontinuous:  $x_i \nearrow x \Rightarrow f(x_i) \nearrow f(x)$

This notion of continuity is equivalent to continuity with respect to the Scott topology:

A subset  $C$  of a dcpo  $X$  is *Scott-closed* if

- (i)  $x \leq y, y \in C \Rightarrow x \in C$
  - (ii)  $(\forall i. x_i \in C)$  and  $x_i \nearrow x \Rightarrow x \in C$ .
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# Continuous dcpos

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For elements  $u$  and  $v$  of a dcpo we say

$u \ll v$  ( $u$  is *way-below*  $v$  or  $u$  is *relatively compact* in  $v$ )

if  $x_i \nearrow x \geq v$  implies  $x_i \geq u$  for some  $i$ .

A dcpo is said to be *continuous* if for every  $v$ , there is a directed family  $u_i \ll v$  such that  $u_i \nearrow v$ .

Example:  $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$  is a continuous dcpo;

$r \ll s$  iff  $r < s$ ;

the Scott-open sets are the intervals  $]r, +\infty]$ .

Example: For a locally compact space  $X$ ,

$\mathcal{O}(X)$ , the lattice of open subsets, is a continuous dcpo,

$U \ll V$  iff there is a compact set  $K$  such that  $U \subseteq K \subseteq V$ .

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# Measures and Valuations

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Measure:  $m : \mathcal{B} \rightarrow R_+$

defined on a Boolean algebra  $\mathcal{B}$  (of subsets of a set  $X$ )

finitely additive:  $A \cap B = \emptyset \Rightarrow m(A \cup B) = m(A) + m(B)$

countably additive:  $A_n \nearrow A \Rightarrow m(A_n) \nearrow m(A)$

in addition to finite additivity provided  $\mathcal{B}$  is a  $\sigma$ -algebra.

Valuation:  $m : \mathcal{L} \rightarrow \overline{R}_+$

defined on a lattice  $\mathcal{L}$  (of subsets of a set  $X$ )

strict:  $m(\emptyset) = 0$

monotone:  $A \subseteq B \Rightarrow m(A) \leq m(B)$

modular:  $m(A \cup B) = m(A) + m(B) - m(A \cap B)$

$m$  is Scott-continuous if  $A_i \nearrow A \Rightarrow m(A_i) \nearrow m(A)$ .

provided  $\mathcal{L}$  is directed complete.

# Measures and Valuations (ctd)

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For Boolean algebras, finitely additive measures and valuations agree.

The notion of a valuation goes back to G. Birkhoff (1939) for arbitrary lattices. Valuations and their relation to measures have been considered by G. Choquet (1955). As a substitute for measures they are a standard tool in Geometric Probability Theory (Schneider, McMullen, Klain, Rota).

# Valuations and Borel Measures

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For a topological space  $X$ , let  $\mathcal{O}(X)$  denote the lattice of open subsets.

The restriction of a Borel measure on  $X$  to the open sets is a valuation on  $\mathcal{O}(X)$ , but not necessarily Scott-continuous.

But: A regular Borel measure on a locally compact Hausdorff space restricts to a Scott-continuous valuation on the opens.

(A Borel measure is inner regular if  $m(K_i) \nearrow m(B)$  when the  $K_i$  range over the compact subsets of the Borel set  $B$ .)

# Valuations and Borel Measures

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Conversely, one may ask, whether a given Scott-continuous valuation on  $\mathcal{O}(X)$  can be extended to a Borel measure.

For example, on a locally compact sober space, every Scott-continuous valuation can be extended to a Borel measure (Keimel-Lawson 2005).

Scott-continuous valuations on  $\mathcal{O}(X)$  replace Borel measures as long as one wants to integrate (semi-) continuous functions, only.

# Choquet Domain Theoretically

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$L$  a given continuous dcpo which is a  $\wedge$ -semilattice,  
 $\mathcal{H}(L)$  the lattice of Scott-closed subsets.

$m: \mathcal{H}(L) \rightarrow \mathbb{R}_+$  a Scott-continuous valuation.

The distribution function  $F: L \rightarrow \mathbb{R}_+$  defined by

$$F(x) = m(\text{cl}(\{x\}))$$

has the following properties:

- (i)  $F$  is strict:  $\inf_{x \in L} F(x) = 0$
- (ii)  $F$  is totally monotone: ....
- (iii)  $F$  is Scott-continuous:  $x_i \nearrow x \Rightarrow F(x_i) \nearrow F(x)$

And every  $F$  with these properties is the distribution of a uniquely defined Scott-continuous valuation on  $\mathcal{H}(X)$ .

# Proof: 1. Discrete step

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The closures of finite subsets  $E \subseteq L$  form a lattice  $\mathcal{B}$ :

$$cl(E) \cup cl(E') = cl(E \cup E'), \quad cl(E) \cap cl(E') = cl(E \wedge E')$$

Given  $F: L \rightarrow \mathbb{R}$  define  $m: \mathcal{B} \rightarrow \mathbb{R}$  by:

$$m(cl(\{u\})) = F(u)$$

$$m(cl(\{u_1, \dots, u_n\})) =$$

$$\sum_i F(u_i) - \sum_{i < j} F(u_i \wedge u_j) + \sum_{i < j < k} F(u_i \wedge u_j \wedge u_k) - + \dots$$

Then  $m$  is a strict modular map on  $\mathcal{B}$ . It is monotone, hence, a valuation if and only if  $F$  is *totally monotone*, that is, iff

$$u \geq u_1, \dots, u_n \implies F(u) \geq m(cl(\{u_1, \dots, u_n\}))$$



# Proof of Modularity

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$\mathcal{X}_A$  denote the characteristic function of  $A \subseteq L$ ,  
 $V$  the real vector space generated by the  $\mathcal{X}_{cl(\{x\})}, x \in L$ .  
 These characteristic functions are linearly independent, hence  
 a basis of the vector space  $V$ . Claim:  $\mathcal{X}_{cl(E)} \in V$ :

$$\begin{aligned} cl(E) &= \bigcup_{u \in E} cl(\{u\}) = X \setminus \bigcap_{u \in E} (X \setminus cl(\{u\})) \\ \mathcal{X}_{cl(E)} &= 1 - \prod_{u \in E} (1 - \mathcal{X}_{cl(\{u\})}) \\ &= \sum_i \mathcal{X}_{cl(\{u_i\})} - \sum_{i < j} \mathcal{X}_{cl(\{u_i \wedge u_j\})} + - \dots \end{aligned}$$

The function  $x \mapsto F(x)$  has a unique linear extension  $F^*: V \rightarrow \mathbb{R}$ . Define  $m(cl(E)) = F^*(\mathcal{X}_{cl(E)})$ . Then  $m$  is a valuation on the lattice  $\mathcal{B}$  (because  $F^*$  is linear), and

$$m(cl(E)) = F^*(\mathcal{X}_{cl(E)}) = \sum_i F(u_i) - \sum_{i < j} F(u_i \wedge u_j) + - \dots$$


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## Proof: 2. Continuous step

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*Basis* of a continuous dcpo  $L$ : a subset  $B$  such that, for every  $v \in L$ , the set  $\{u \in B \mid u \ll v\}$  is directed and

$$v = \sup_{u \ll v, u \in B} u$$

**Basis Lemma** For any monotone map  $m: B \rightarrow \overline{\mathbb{R}}$ , the map  $m^*: L \rightarrow \overline{\mathbb{R}}$  defined by

$$m^*(v) = \sup_{u \ll v, u \in B} m(u)$$

is Scott-continuous. If  $m^*(u) = m(u)$  for all  $u \in B$ , then  $m^*$  is the unique Scott-continuous extension of  $m$ .

## Proof: Continuous step (ctd)

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**Lemma** For a continuous dcpo  $L$  the set  $\mathcal{H}(L)$  of all closed subsets is a continuous lattice; the lattice  $\mathcal{B}$  of closures  $cl(E)$  of finite subsets  $E$  is a basis  $\mathcal{B}$  and  $cl(E) \ll C$  iff for each  $u \in E$  there is a  $v \in C$  with  $u \ll v$ .

Applying the Basis Lemma we obtain a Scott-continuous  $m^*: \mathcal{H}(L) \rightarrow \mathbb{R}$  by putting

$$m^*(C) = \sup\{m(cl(E)) \mid E \subseteq L \text{ finite}, cl(E) \ll C\}.$$

A continuity argument shows that  $m^*$  is a valuation.

**Technical Lemma** If  $F: L \rightarrow \mathbb{R}$  is Scott-continuous, then  $m(cl(E)) = m^*(cl(E))$  for every finite subset  $E$  of  $L$ .

This finishes the proof.

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# Back to Choquet

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$X$  a locally compact space,  
 $L = \mathcal{O}(X)$ . Our theorem yields a bijection between  
strict, totally monotone Scott-continuous  $F: \mathcal{O}(X) \rightarrow \mathbb{R}$   
and  
Scott-continuous valuations on  $\mathcal{H}(\mathcal{O}(X)) \cong \mathcal{O}(\mathcal{K}(X))$  as is  
not difficult to see.

Our result includes locally compact spaces that need not be Hausdorff provided one chooses the appropriate definition of local compactness (every point has a neighborhood basis of compact neighborhoods) and the appropriate class of compact sets (compact saturated sets).

# Conclusion

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I am sure that I have convinced you that domain theoretical ideas are useful to deal with classical arguments in analysis as far as they are based on compactness arguments and to extend them to non Hausdorff situations.

We are just writing down the Choquet type theorems for the usual powerdomains in semantics: demonic, angelic, erratic.

The construction contains more potential to be exploited:

- The continuous domains need not be directed complete.
- Integrals of lower semicontinuous functions should be directly definable by a completion of the vector space  $V$  generated by the characteristic functions.