Degrees of co-c.e. closed sets with specific computability-theoretic properties

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Motivation

1. From viewpoint of **Recursion Theory**:
   Co-c.e. closed sets (lightface Π^0_1 classes) in 2^ω or ω^ω are the first level in the arithmetical hierarchy which can have no computable points. So, study of Turing degrees of points of co-c.e. closed sets seems to be significant as that of Turing degrees of c.e. sets in $\mathbb{N}$.
2. From viewpoint of **Reverse Mathematics**:
   The existence of points of a nonempty co-c.e. closed set is an instance of **WKL**. Study of degrees of co-c.e. closed sets helps us to observe what kind of instance of **WKL** can occur.

Co-c.e. closed sets

1. S: a computable metric space with a numerated basic open balls $\{B_i\}_{i\in\omega}$.
2. We say that a closed set C in S is co-c.e. if $C = S - \bigcup_{i\in W} B_i$ for some c.e. set W.
3. In 2^ω or ω^ω, co-c.e. closed sets coincide with lightface Π^0_1 definable sets.
4. In 2^ω or ω^ω,
   1. a co-c.e. closed set is represented by infinite paths of a computable tree;
   2. infinite paths of a computable tree form a co-c.e. closed set.
5. Degrees of points of co-c.e. closed sets (Π^0_1 classes) are first studied by Kleene (1943).

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   is co-c.e. closed in $\mathbf{2}^\omega$.
3. The set of all zeros of computable function on $\mathbb{R}$ is co-c.e. closed in $\mathbb{R}$.
4. The Mandelbrot set is co-c.e. closed in $\mathbb{C}$.

A closed set in a computable metric space is special if it is nonempty and contains no computable points.

Special closed sets

1. Krisel (1953): A special co-c.e. closed set exists in $\mathbf{2}^\omega$.
2. Lacombe (1956): A special co-c.e. closed set exists in $\mathbb{R}$.

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3. Shoenfield (1960): The set of all complete consistent extensions of a c.e. theory forms a co-c.e. closed set in $\mathbf{2}^\omega$.
4. So the existence of special co-c.e. closed sets is also derived from the following two theorems:
    1. (Gödel’s incompleteness theorem) Elementary arithmetic has no complete consistent c.e. extensions.
    2. (Lindenbaum’s lemma) Every consistent theory has a complete consistent extension.

Some nonempty co-c.e. closed sets contain no computable points, but in certain space, they must contain some points of low-complexity.

Definition

- A is limit-computable if $A \leq_T \mathbf{0}'$.
- A is low if $A' \equiv_T \mathbf{0}'$.

(where A' denotes the Turing jump of A, and $\mathbf{0}'$ denotes the halting problem.)

Basis Theorems

1. Kreisel (1958): every nonempty co-c.e. closed set in $\mathbf{2}^\omega$ contains a limit-computable point.
2. Shoenfield (1960): every nonempty co-c.e. closed set in $\mathbf{2}^\omega$ contains a point $<_T \mathbf{0}'$.

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Low Basis Theorems

1. Jockusch/Soare (1972): every nonempty co-c.e. closed set in $\mathbf{2}^\omega$ contains a low point.
2. Brattka/de Brecht/Pauly (2010): every nonempty co-c.e. closed set in a computable σ-compact space contains a low point. (For example, every nonempty co-c.e. closed set in $\mathbb{R}$ has a low point.)

We now define main tools to analyze degrees of points of closed sets.

Definition

1. (Medvedev, 1955) $P \leq_s Q$ if there is a computable map from Q to P.
2. (Muchnik, 1963) $P \leq_w Q$ if every point of Q computes a point of P.

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Intuition

1. $P \leq_s Q$ (via f): If we get a solution x to Q then we also get a solution $f(x)$ to P.
2. $P \leq_w Q$: If Q has a solution x then P has more easier solution $y \leq_T x$.

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Intuition

1. $P \leq_s Q$: the statement "the tree T_Q has a path" is stronger than "the tree T_P has a path", and this is witnessed by a computable way.
2. $P \leq_w Q$: the statement "the tree T_Q has a path" is stronger than "the tree T_P has a path".

1. $\mathcal{P}_s = $ (all nonempty co-c.e. closed sets in $\mathbf{2}^\omega$)$/\equiv_s$.
2. $\mathcal{P}_w = $ (all nonempty co-c.e. closed sets in $\mathbf{2}^\omega$)$/\equiv_w$.

We can think of these degree notions as representing the strength of an instance of **WKL**.

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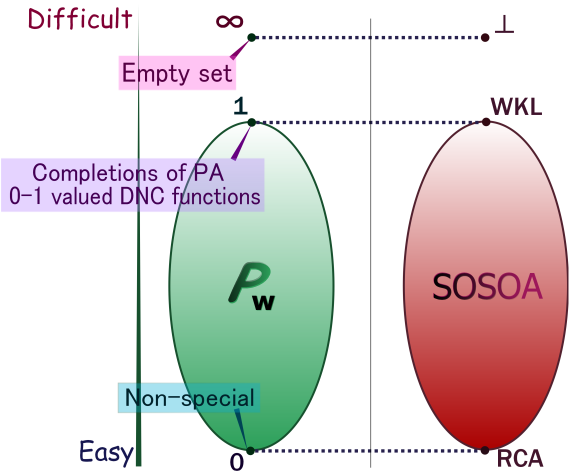
2. $\mathcal{P}_w = (\text{all nonempty co-c.e. closed sets in } \mathbf{2}^\omega)/\equiv_w$.

We can think of these degree notions as representing the strength of an instance of **WKL**.

1. **the Medvedev degrees** is closely related to **computable reverse analysis** (Brattka)
   - Brattka-Gherardi (2009) showed that the Medvedev lattice is embedded into the Weihrauch lattice.
2. **the Muchnik degrees** is closely related to **classical reverse mathematics** (Friedman-Simpson)
   - Mummert (2008) get some embedding theorems about the Lindenbaum algebra $\mathcal{L}(\mathbf{WKL_0}, \mathbf{RCA_0})$ of second-order arithmetic between $\mathbf{WKL_0}$ and $\mathbf{RCA_0}$ by applying Binns-Simpson embedding theorem for the Muchnik degrees of co-c.e. closed sets (2003).


Difficult
∞
⊥
Empty set
1
WKL
Completions of PA
0–1 valued DNC functions
$\mathcal{P}_w$
SOSOA
Non-special
Easy
0
RCA

Previous Works

First we explain our previous research for degrees of co-c.e. closed sets.

Now, we focus on degree structures of co-c.e. closed sets in 2^ω.

1. $[\sigma]$ denotes the clopen set $\{f \in 2^{\mathbb{N}} : f \supset \sigma\}$.
2. Tree representation for a closed set P:
   $T_P = \{\sigma \in 2^{<\omega} : [\sigma] \cap P \neq \emptyset\}$.
3. Let V_P be a computable tree for which all paths and a closed set P coincides, and V_P^{ext} denotes all extendible nodes of V_P. Then $V_P^{ext} = T_P$.

Our previous works are based on the Medvedev degree structure of co-c.e. closed sets in $\mathbf{2}^\omega$.

Theorem by Cole (in PhD thesis, 2009); Kihara (in Master thesis, 2009); Cole-Kihara (2010)

The $\forall\exists$-theory of the Medvedev degrees of co-c.e. closed sets in $\mathbf{2}^\omega$ is decidable in the language $\mathcal{L} = \{\leq, \mathbf{0}, \mathbf{1}\}$.

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We also want to decide the structure of Muchnik degrees. However, the Muchnik degree structure seems to be much harder to study than the Medvedev degree structure. For example, it is open whether or not even the following simple $\forall\exists$-sentence is true:

$$\mathcal{P}_w \models (\forall a < 1)(\exists b < 1)\ a < b.$$

In contrast, we have a decision procedure determining the $\forall\exists$-truth of $\mathcal{P}_s$ as described above.

1. One of the reason of the difficulty is that a Muchnik reduction might be heavily discontinuous, whereas a Medvedev reduction is necessarily continuous.
2. However, we notice that both of these reductions are *computably invariant*, in the sense of Brattka (1999).
3. Indeed, for Medvedev reduction or Muchnik reduction f, clearly the following holds:
   - $(\forall x \in \mathrm{dom}(f))\ f(x) \leq_T x$.

On our previous work, we introduce some computably invariant reductions between Muchnik reduction and Medvedev reduction to study degrees of points of co-c.e. closed sets.

↑ Discontinuous; ↓ Continuous:

- Muchnik reduction (Muchnik, 1963),
- *Learning*-reduction by *a team*,
- *Learning*-reduction, (based on the notion "identifications in limit" (Gold, 1967); equivalent to ρ_H-computable function (Ziegler, and others)),
- *Learning*-reduction with *bounded errors*,
- *Learning*-reduction with *bounded mind-changes*,
- Para-computable function (Yasugi-Tsujii, 2005; $\mathcal{L}$-sequentially computable $+$ piecewise effectively continuous),
- Medvedev reduction (computable function).

New Works: Degrees of immune closed sets

In this part, we give a new result about degrees of the specific co-c.e. closed set. We first focus on immunity for closed sets.

This part is included in the joint work with Douglas Cenzer, Rebecca Weber, and Guohua Wu.

A closed set is immune if its tree representation has no infinite c.e. subtree.

Application of immunity I

Demuth-Kucera (1987) used immunity of co-c.e. closed sets in ω^ω to study relationship between **1**-generic and **1**-randomness (Martin-Löf randomness).

- Every **1**-generic computes no point in any immune co-c.e. closed sets in ω^ω.
- The set of all fixed-point-free functions forms an immune co-c.e. closed set in ω^ω.

These result implies the following:

- Every **1**-generic real computes no **1**-random real.

Application of immunity II

It is known that every c.e. closed sets in computable Polish space is effectively separable (they actually coincides). So it is natural to ask whether it also holds in any *incomplete* computable metric space. By using notions of immunity, Brattka (2002) showed the following:

- In some incomplete computable metric space (e.g. $\mathbb{Q}$), there exists a c.e. closed which is *not* effectively separable.

Now, we apply some variant of immunity to resolve an analogy of Post's problem. Recall that the original Post's problem is the following:

Post's Problem (1944)

Find an intermediate c.e. Turing degree!

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3. Harrington and Soare (1987) provides a $\mathcal{E}$-definable solution to Post's problem.
4. (Question) Find a concrete solution to Post's problem!

Analogy of Post's Problem

Find a concrete natural intermediate degree in $\mathcal{P}_w$!

We do not mention whether above examples are natural or not.

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Find a concrete natural intermediate degree in $\mathcal{P}_w$!

Theorem (Simpson)

1. *(Simpson's Embeddings Theorem, 2007) For every lightface $\boldsymbol{\Sigma}^0_3$ set $S \subseteq \mathbb{N}^{\mathbb{N}}$ and every co-c.e. closed set $P \subseteq 2^\omega$, the Muchnik degree of $P \cup S$ belongs to $\mathcal{P}_w$.*

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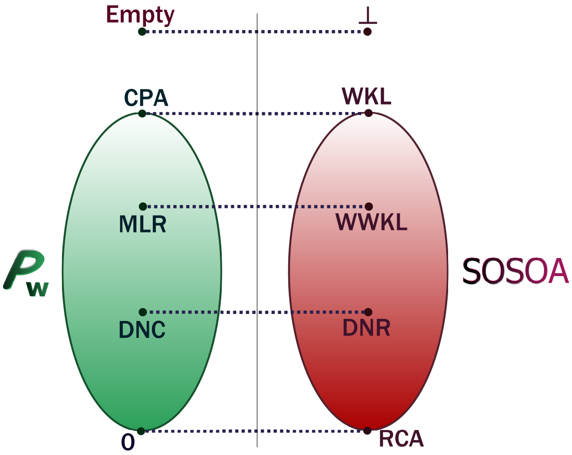
2. *The Muchnik degree of all diagonally noncomputable functions* **DNC** *is intermediate in $\mathcal{P}_w$.*
3. *The Muchnik degree of all Martin-Löf random reals* **MLR** *is intermediate in $\mathcal{P}_w$. This degree is characterized by the greatest Muchnik degree of co-c.e. closed sets of Lebesgue measure > 0.*

We do not mention whether above examples are natural or not.


Empty
⊥
CPA
WKL
MLR
WWKL
$\mathcal{P}_w$
SOSOA
DNC
DNR
0
RCA

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Find a concrete natural intermediate degree in $\mathcal{P}_s$!

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2. **DNC** is co-c.e. closed in ω^ω but not in $\mathbf{2}^\omega$.
3. (Terwijn, 2006) Their Muchnik degrees are contained in $\mathcal{P}_w$, but neither of them is contained in $\mathcal{P}_s$.

Definition (Cenzer-Kihara-Weber-Wu)

A closed set P is tree-immune if there is no infinite computable tree V_P (which might have dead ends) such that $V_P \subseteq T_P$.

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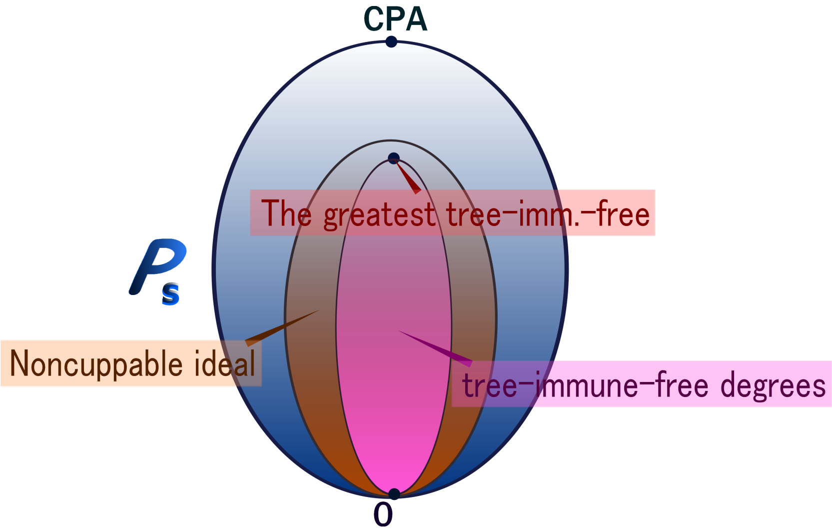
The greatest Medvedev degrees without tree-immune co-c.e. set exists, and it has an intermediate degree in $\mathcal{P}_s$.

Lemma

1. *If $\mathbf{a}$ contains no tree-immune set, then $\mathbf{a}$ is noncuppable.*
2. *If $\mathbf{a}$ contains a non-tree-immune set, then $\mathbf{a}$ contains no tree-immune set.*
3. *Let $\overline{\mathbf{TIM}} \subseteq \mathcal{P}_s$ denote all degrees containing no tree-immune co-c.e. closed sets, then $\mathbf{max}\,\overline{\mathbf{TIM}}$ exists.*


CPA
The greatest tree-imm.-free
$\mathcal{P}_s$
Noncuppable ideal
tree-immune-free degrees
0

The following notion is important to conclude the first lemma.

1. A set $H \subseteq 2^\omega$ is homogeneous if $H = \prod_n F_n$ for some sequence $\{F_n\}$, where $F_n \subseteq \{0, 1\}$ for any n.
2. Clearly, a homogeneous set is co-c.e. closed.
3. DNC_2 is a homogeneous set of Medvedev degree $\mathbf{1}$.

Lemma

If $\mathbf{a}$ contains no tree-immune set, then $\mathbf{a}$ is noncuppable, that is, $\mathcal{P}_s \models (\forall \mathbf{b} < \mathbf{1})\ \mathbf{a} \vee \mathbf{b} < \mathbf{1}$.

Proof.

- Pick a homogeneous set $H = \prod_n F_n \in \mathbf{1}$.
- Every $P \in \mathbf{a}$ is non-tree-immune, so we have $V_P \subseteq T_P$.
- Assume $H \leq_s P \vee Q \in \mathbf{a} \vee \mathbf{b}$ via a computable map Φ.
- For given n and $f \in Q$, we computes the least $\sigma \in V_P$ for which $\Phi(\sigma \oplus f; n) \downarrow = k$ for some k, and let $\Psi(f; n) = k$.
- Clearly, $\Phi(f; n) \in F_n$ for every n.
- Then $H = \prod_n F_n \leq_s Q$ via Ψ, i.e. $\mathbf{b} = \mathbf{1}$.

Lemma

If $\mathbf{a}$ contains a non-tree-immune co-c.e. closed set, then $\mathbf{a}$ contains no tree-immune co-c.e. closed sets.

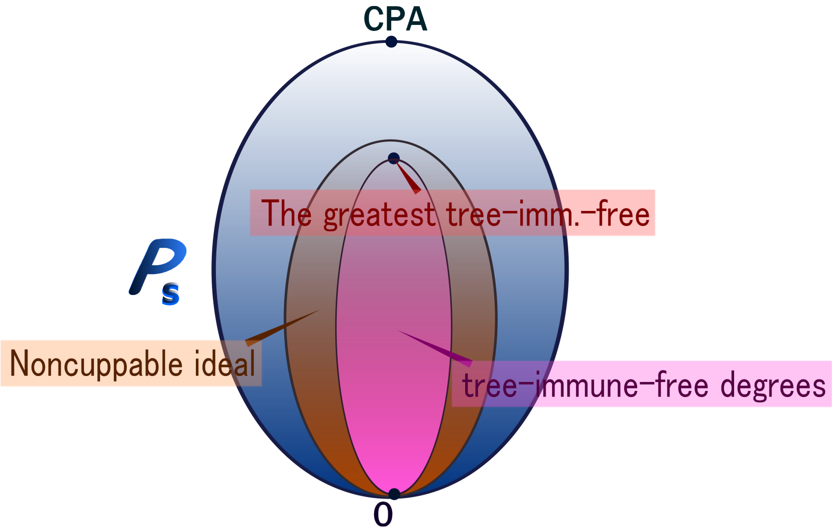
Proof.

- Assume $P, Q \in \mathbf{a}$, and P is non-tree-immune.
- $\Phi : P \to Q$: a computable map.
- V_P: an infinite computable tree.
- $\Phi(V_P) \subseteq T_Q$ is clearly a computable tree.
- For $f \in [V_P] \subseteq P$, we have $\Phi(f) \in Q$ since $Q \leq_s P$.
- Thus $\Phi(V_P)$ has a path $\Phi(f)$, and so $\Phi(V_P) \subseteq T_Q$ is infinite.

Lemma

Let $\overline{TIM} \subseteq \mathcal{P}_s$ denote all degrees containing no tree-immune co-c.e. closed sets, then $\max \overline{TIM}$ exists.

- $M = \bigcup_{\sigma \in L(CPA)} \sigma CPA$ is a desired one.
- P: a non-tree-immune co-c.e. closed set.
- By using $\Phi : CPA \to [V_P]$, define $\Phi^* : CPA \to [V_P]$ s.t. $\Phi^*(L(CPA)) \subseteq L(V_P)$.
- We need to define a computable map $\Psi : M \to P$.
- $\Psi(f) = \Phi^*(f)$ for $f \in CPA$.
- For $f \notin CPA$,
  - Some $\rho \subset f$ belongs to $L(CPA)$.
  - So we compute an index e of $P(\supseteq \Phi^*(\rho))$.
  - $\Psi(f) = \Delta_e(f)$, where $P_e \leq_s CPA$ via Δ_e.


CPA
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New Works: Degrees of K-trivial closed sets, and incompletely generated closed sets.

1. Martin-Löf randomness for closed sets is introduced by Broadhead-Cenzer-Dashti (2006).
2. (Fact) A real z is Martin-Löf random iff $K(z \upharpoonright n) \geq n - O(1)$, where K denotes the prefix-free Kolmogorov complexity.
3. Chaitin defined a real z to be K-trivial if $K(z \upharpoonright n) \leq K(0^n) + O(1)$.
4. K-triviality for closed sets is introduced by Barmpalias-Cenzer-Remmel-Weber (2009).
5. A closed set A is K-trivial if its tree representation is K-trivial.

Theorem (Barmpalias-Cenzer-Remmel-Weber)

Muchnik (Medvedev) degrees of $\boldsymbol{K}$-trivial co-c.e. closed sets are upward dense.

Theorem (Barmpalias-Cenzer-Remmel-Weber)

Medvedev degrees of $\boldsymbol{K}$-trivial homogeneous sets are dense.

Problem (Barmpalias-Cenzer-Remmel-Weber)

Are Medvedev degrees of $\boldsymbol{K}$-trivial co-c.e. closed sets dense?

Theorem

Medvedev degrees of K-trivial co-c.e. closed sets are dense.

Idea of the proof.

1. K-triviality is invariant under tt-preserving maps.
2. Every co-c.e. closed set $P \subseteq 2^\omega$ is Medvedev-bounded by a homogeneous set $H = \prod_n F_n$ of $T_P \equiv_{tt} \bigoplus_n F_n$. (The proof is similar to that of $\mathbf{RCA_0} \vdash \Sigma^0_1$-SEP$\rightarrow$ $\mathbf{WKL}$.)
3. By applying Binns' splitting theorem to K-trivial sets, we obtain an affirmative answer to the above problem.

Upward density problem for $\mathcal{P}_w$

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As seen before, it is true for K-trivial co-c.e. closed sets.
However, the class of K-trivials are very narrow.
We should consider the above problem for more wider class:

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We should consider the above problem for more wider class:

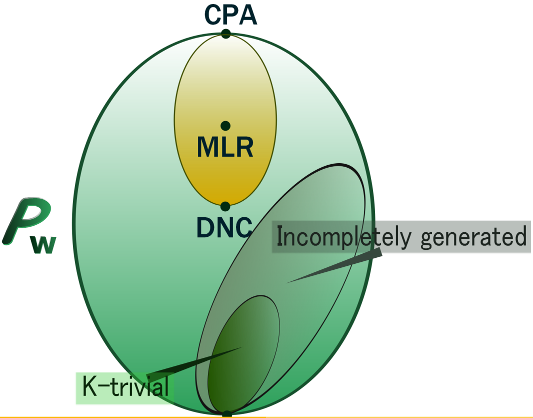
Definition

A co-c.e. closed set is incompletely generated (i.g.) if its tree representation is $<_T \emptyset'$.

proposition

Let $\boldsymbol{P}$ be a nonempty co-c.e. closed set in $\mathbf{2}^\omega$;

1. $\boldsymbol{P}$ is $\boldsymbol{K}$-trivial $\Rightarrow$ $\boldsymbol{P}$ is incompletely generated.
2. $\boldsymbol{P}$ is incompletely generated $\Rightarrow$ $\mathbf{DNC} \not\leq_w \boldsymbol{P}$.



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There exists a Muchnik (Medvedev) degree such that

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Key lemmas (we omit the proof in this talk.)

1. There exists an i.g. co-c.e. closed Medvedev degree bounding all Medvedev degrees of K-trivial co-c.e. closed sets.
2. For every c.e. set $A <_T \mathbf{0}'$, there exists an i.g. co-c.e. closed set without A-computable points.

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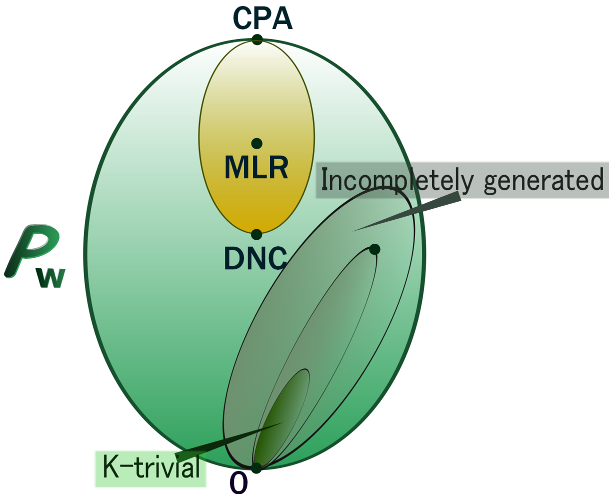
Theorem

1. Muchnik degrees of i.g. co-c.e. closed sets are upward dense.
2. Medvedev degrees of i.g. co-c.e. closed sets are dense.


CPA
MLR
DNC
Incompletely generated
K-trivial
0
Pw

Conclusion

From viewpoint of **Recursion Theory**:

1. (An answer to Post's problem for $\mathcal{P}_s$) We find an intermediate co-c.e. closed Medvedev degree which can be defined by using only a combinatorial property for closed sets.
2. (Partial solution to upward density problem for $\mathcal{P}_w$) I.g. co-c.e. closed Muchnik degrees are upward dense.

From viewpoints of **Reverse Mathematics** and **Computable Analysis**:

- Instances of **WKL** occurs densely, even if we restrict it to K-trivial instances, or i.g. instances.

Thank you!