The role of unique choice
in the minimalist foundation

j.w.w. *G. Sambin*

Maria Emilia Maietti
University of Padova

Role of unique choice

In our minimalist foundation [MS'05], [M'09]

two distinct notions of function $\begin{cases} \text{functional relation} \\ \text{operation (or type-theoretic function)} \end{cases}$

Axiom of unique choice : functional relations = operations

Why NO unique choice: Doing without Turing machines

In our minimalist foundation [MS'05], [M'09]

operation= lawlike function = computable function

$$\neq$$

functional relation = generic function

Why NO unique choice

In our minimalist foundation [MS'05], [M'09]

operation = lawlike function = computable function

$\Rightarrow$ Op(A,B)=operations from A set to B set

form a SET

functional relation = generic function

$\Rightarrow$ Funrel(A, B) = functional relations from A set to B set

do NOT form a set

Why no unique choice

our minimalist foundation [MS'05], [M'09]

is compatible with Feferman's theory of explicit mathematics ('79)

even with its CLASSICAL PREDICATIVE version.

instead Aczel's CZF+ EM = ZF

Starting issue

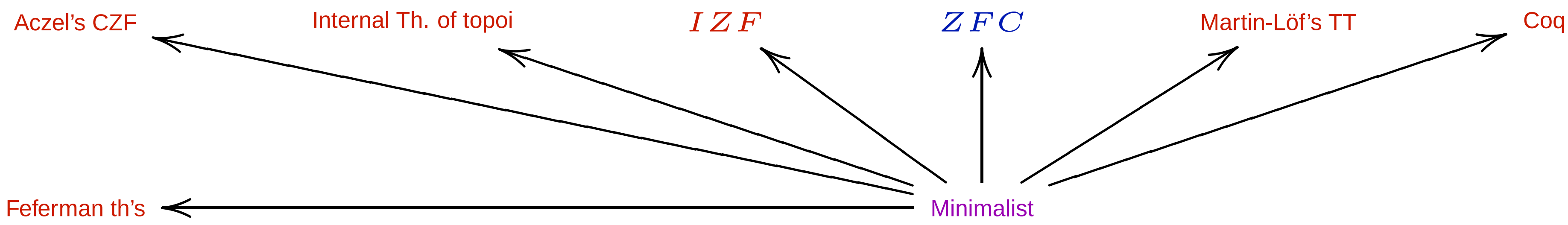
Our minimalist foundation [MS'05], [M'09]

is an answer to:

Is there a minimalist foundation

compatible with the most relevant constructive foundations

and then *also* with classical set theory?



minimalist foundation must be predicative!

Why not HA^{Ω} *as minimalist foundation?*

HA^{Ω} = Heyting arithmetics with finite types

NOT sufficient

to represent predicative power collection of subsets

(and constructive topology ....)

$\Rightarrow$ need of two distinct notions: sets/collections (or sets/classes)

What is a constructive foundation?

Informal definition

a theory is **constructive**

= it has a realizability model where to extract programs from proofs.

What is a constructive foundation?

from [MS'05]

Formal Assumption: a theory is **constructive**,

- called a **proofs-as-programs** theory -

if consistent

with Axiom of choice AC + formal Church thesis CT

Idea motivating this definition: think of HA^{Ω} realized by

Kleene's realizability interpretation

Axiom of choice

$$(AC) \qquad \forall x \in A \ \exists y \in B \ R(x,y) \quad \longrightarrow \quad \exists f \in A \to B \ \ \forall x \in A \ \ R(\,x\,,\ f(x)\,)$$

from any total relation we can extract a function

Formal Church thesis

$$(CT) \qquad \forall f \in \mathsf{N}at \longrightarrow \mathsf{N}at \quad \exists e \in \mathsf{N}at$$

$$(\, \forall x \in \mathsf{N}at \;\; \exists y \in \mathsf{N}at \;\; T(e, x, y) \;\&\; U(y) =_{\mathsf{N}at} f(x) \,)$$

every function from $\mathsf{N}at$ to $\mathsf{N}at$ is internally recursive

Extraction of programs from proofs

Axiom of choice AC + formal Church thesis CT

means

from any specification as total relation we can extract a function
that is recursive if from Nat to Nat

*Too narrow Definition of **constructive** theory?*

Our Definition of **proofs-as-programs** theory

= theory **consistent** with

Axiom of choice AC+ formal Church thesis CT

is very technical

but no contraexample found yet......

Problem: Is there a relevant commonly conceived proofs-as-programs theory

NOT CONSISTENT with CT + AC?

Usefulness of our **proofs-as-programs** *definition*

Our notion of **proofs-as-programs** theory is very useful to discriminate theories:

- constructive versus classical theories

$$\text{Peano Arithmetics} + \text{CT} + \text{AC} \vdash 0 = 1$$

CT + AC! sufficies for this

CT + AC $\Rightarrow$ Extended Church thesis

- intensional versus extensional theories

$$\text{Heyting arithmetics with finite types} + \text{CT} + \text{AC} + \text{extfun} \vdash \perp$$

$$\text{extfun}\ \frac{f(x) =_B g(x)\ \ true\ \ [x \in A]}{\lambda x.f(x) =_{A \to B} \lambda x.g(x)\ \ true} \qquad \text{extensionality of functions}$$

Notion of TWO-LEVEL constructive foundation

from [MS'05]

A foundation for constructive mathematics is a two-level theory with

- an **intensional** level that is a PROOFS-AS-PROGRAMS theory *to be meant as a programming language*

- an **extensional** level ABSTRACTION of the intensional one
  via Sambin's *forget-restore principle* including
  quotients
  extensionality of operations/functions
  proof-uniqueness of propositions

current work j.w.w. G. Rosolini:
characterize the link between the levels (categorically).

What language to choose for the foundation?

as an *AXIOMATIC SET THEORY*: gives global axioms of set existence

(mainly extensional with implicit existence)

usefulness: formalization of math proofs as in common practice

suitable to descrive extensional level- future work

as a *CATEGORY*: gives the algebraic structure of models for the intended theory

(mainly extensional via universal properties)

usefulness: to single out the same structure in different contexts /unification of structures

suitable to describe the link between levels - future work

as a *TYPE THEORY*: gives computational contents of set constructions

(mainly intensional with explicit existence of sets).

usefulness: extraction of programs from proofs

suitable to describe the intensional level

Our two level minimalist constructive foundation

from [M'09]

- its intensional level= intensional type theory à la Martin-Löf

  = many sorted logic with propositions + proof-terms

  with 4 sorts:
  - collections, sets
  - propositions (seen as collections of their proofs)
  - small propositions (seen as sets of their proofs)

  it is a PREDICATIVE VERSION of Coquand's Calculus of Constructions (Coq).

- its extensional level= emTT extensional type theory à la Martin-Löf

  = many sorted logic

  + proof uniqueness of propositions

  + *effective* quotient sets

  + *extensionality* of typed functions

Bishop's pre-sets = sets in our intensional level

Bishop's sets = setoids in our intensional level= sets in our extensional level

What links the two levels

in [M'09]:

to interpret the extensional level over the intensional one

we built a quotient model over the intensional level

based on Bishop's total setoids...

BENEFIT of two levels: we do NOT develop math proofs WITHIN the setoid model directly !!!

Benefits of our foundation: No unique choice

elimination rule of propositions ONLY toward propositions.

$\Longrightarrow$ NO axiom of unique choice

$\Longrightarrow$NO axiom of choice

BOTH at our intensional level and extensional level

$\Longrightarrow$

two distinct notions of functions as Feferman's distinction function/operation

$$\begin{cases} \text{functional relation} & \forall x \in A\ \exists! y \in B\ R(x,y) \\ \\ \text{type-theoretic function (=operation)} & f \in A \rightarrow B \\ & \text{given by terms } f(x) \in B\ [x \in A] \end{cases}$$

Axiom of unique choice

(AC!) $\forall x \in A \, \exists! y \in B \, R(x, y) \;\longrightarrow\; \exists f \in A \to B \, \forall x \in A \, R(x, f(x))$

turns a functional relation into a type-theoretic function/operation.

$\Rightarrow$ identifies the two distinct notions...

Only operations form a set in our foundation

at both levels of our foundation

$$\mathsf{Op}(A, B) \equiv A \rightarrow B$$

operations from A set to B set form a SET

$$\mathsf{Funrel}(\mathrm{A}, \mathrm{B}) \equiv \{ \mathrm{R} \in \mathcal{P}(\mathrm{A} \times \mathrm{B}) \mid \mathrm{R} \text{ functional relation } \}$$

collection of functional relations from A set to B set is NOT generally a set

$\Rightarrow$ compatibility with Feferman's classical predicative theories.

Different axiom of choices

different axiom of choices:

AC$_{funrel}$ = from a total relation extract a functional relation

AC$_{\lambda}$ = from a total relation extract a type-theoretic function (or operation)

$$\text{AC}_{\lambda} \quad \Rightarrow \quad \text{AC}_{funrel}$$

AC!=AC!$_{\lambda}$ = from a functional relation extract a type-theoretic function (or operation)

AC!$_{funrel}$=from a functional relation extract a functional relation tautology!!

WARNING on the two levels

different unique choices:

AC! at our intensional level

$$\neq$$

AC! at our extensional level

different axioms of choice:

AC_λ at our intensional level

$$\neq$$

AC_λ at our extensional level

WARNING on the two levels

AC_λ at our intensional level $\quad\Rightarrow\quad$ AC! at our extensional level

WARNING on the two levels

AC_{λ} at our extensional level $\Rightarrow$ excluded middle

is NOT CONSTRUCTIVE

AC_{λ} at our intensional level $\Leftrightarrow$ Aczel's presentation axiom

in our quotient model over the intensional level

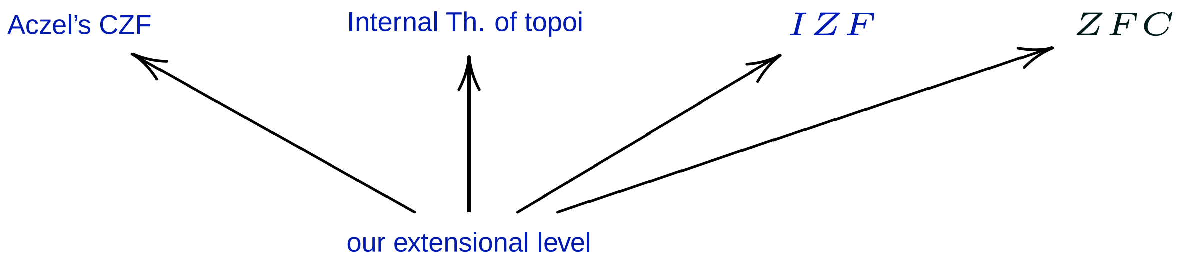
is CONSTRUCTIVE

Problem: what is the type-theoretic formulation of Aczel's presentation axiom

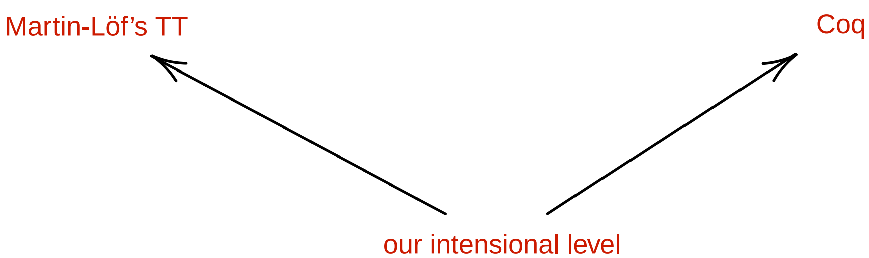
in our extensional theory??

WARNING on compatibility

- compatibility with extensional theories is with our extensional level



- compatibility with intensional theories is with our intensional level



$\Rightarrow$ do NOT compare directly Martin-Löf's TT with Internal Th. of topoi

BUT compare the internal theory of its quotient model with it

Benefits of our minimalist foundation

same benefits as Heyting arithmetics HA^{Ω}

our extensional foundation consistent with

AC! +CT	CT_{λ} + NO AC!	NO CT_{λ} + AC!
Funrel(Nat,Nat)= Op(Nat,Nat)	Op(Nat,Nat)= recursive functions	
all functional relations are recursive	NOT all functional relations are recursive	NOT all operations are recursive
quotient model on extended Kleene's realizability	*quotient model on that in* [M'10]	*usual interpretation* in ZFC

Benefits of our minimalist foundation

same benefits as Heyting arithmetics HA(Ω)

our intensional foundation consistent with

AC_λ +CT	CT_λ + NO AC!	NO CT_λ + AC!
Funrel(Nat,Nat)= Op(Nat,Nat)	Op(Nat,Nat)= recursive functions	
all functional relations are recursive	NOT all functional relations are recursive	NOT all operations are recursive
extension of *Kleene's realizability* $\Rightarrow$ ours is a proofs-as-programs theory	*model in* [M'10]	*usual interpretation* in ZFC

Benefits of our minimalist foundation

same benefits as Heyting arithmetics HA^{Ω}

our extensional foundation consistent with

AC! +CT	CT_{λ} + NO AC!	NO CT_{λ} + AC!
NO Bar Induction	+ Bar Induction	+ Bar Induction
Funrel(Nat,Nat)= Op(Nat,Nat)	Op(Nat,Nat)= recursive functions	
all functional relations are recursive	NOT all functional relations are recursive	NOT all operations are recursive
quotient model on *extended Kleene's realizability*	*quotient model on* *that in* [M'10]	*usual interpretation* in ZFC

Traditional Bar Induction

traditional Bar Induction:

if $Q \subset$ List(Nat), Q inductive

V monotone bar of the empty list nil

$V \subset Q$

$\Rightarrow$ nilεQ

V is a bar of nil

if every choice sequence goes through an element in V

i.e. $\forall \alpha$ choice sequence $\exists\, v \varepsilon V \;\; v = \lfloor \alpha(0), \ldots, \alpha(n) \rfloor$

$\Rightarrow$ DEPENDS on the notion of choice sequence

Bar Induction as spatiality

from [Fourman-Grayson'82, Sambin'87, Gambino-Schuster'07, Sambin'08]

traditional Bar Induction = spatiality of pointfree Baire formal topology
called BI(Nat)

traditional Fan theorem Fan = spatiality of pointfree Cantor formal topology
called BI($\{0, 1\}$)

if choice sequences = functional relations from Nat to Nat

because
notion of formal point = functional relation

Kleene's result in the literature

from [Troelstra-van Dalen'88]

$$\mathrm{HA}^{\omega} + \mathrm{Fan}_{\lambda} + \mathrm{CT}_{\lambda} \vdash \bot$$

Fan_{λ} = Fan theorem with choice sequences as type-theoretic functions

but since

$$\mathrm{Fan} + \mathrm{AC!} \quad \Rightarrow \quad \mathrm{Fan}_{\lambda}$$

Kleene's result $\Rightarrow$ our extensional foundation $+$ Fan $+$ $\mathrm{AC!}$ $+$ $\mathrm{CT}_{\lambda} \vdash \bot$

Kleene's result in the literature

Kleene's result in an *axiomatic set theory*:

$$\mathsf{CZF} + \mathsf{Fan} + \mathsf{CT}_{funrel} \vdash \bot$$

with choice sequences= functional relations

CT_{funrel} = CT for functional relations

$\Rightarrow$ also becomes

$$\text{our extensional foundation} + \mathsf{Fan} + \mathsf{AC!} + \mathsf{CT}_{\lambda} \vdash \bot$$

since

$$\mathsf{AC!} + \mathsf{CT}_{\lambda} \Rightarrow \mathsf{CT}_{funrel}$$

Bar Induction on generic tree

for any set A

BI(A) $=$ Bar Induction on the tree List(A)

choice sequences= functional relations from Nat to A

$=$ spatiality of the formal topology on the tree List(A)

Consistency with Bar Induction + CT_λ

supposing ZF+ DC (axiom of dependent choice) consistent

$\Downarrow$

our extensional level is CONSISTENT with BI(A) + CT_λ

(+ existence of inductively generated formal topology put on tree List(A) for A set)

Consistency with Bar Induction + CT_λ

it is enough to prove

our INTENSIONAL level is CONSISTENT with $\mathrm{BI(A)}^i$ + CT^i_λ

$\mathrm{BI(A)}^i$ = translation of BI(A) at the intensional level

CT^i_λ = translation of CT_λ at the intensional level

via a realizability model in ZF+DC :

- interpret our sets as subsets of natural numbers via Kleene realizability
- interpret our propositions as their boolean value
- interpret our proper collections as ZF-sets

No unique choice in our foundation

our extensional level $+$ Fan $+$ CT_λ $+$ AC! $\vdash \bot$

$+$

CONSISTENCY of our extensional level + Fan + CT_λ

$\Downarrow$

no AC! in our extensional level

Inductively generated Baire formal topology

basic opens indexed by List(Nat)

collection of *formal opens* = fix-points of (stable) closure operator $\mathcal{B}aire$

[Fourman-Grayson'82, Sambin'87]

for subset V of List(Nat)

$\mathcal{B}aire(V)$ generated from axioms as in [CSSV'03]

$$\frac{l \varepsilon V}{l \varepsilon \mathcal{B}aire(V)} \qquad \frac{s = \lfloor l, t \rfloor \quad l \varepsilon \mathcal{B}aire(V)}{s \varepsilon \mathcal{B}aire(V)}$$

$$\frac{\forall n \in Nat \ \ \lfloor l, n \rfloor \varepsilon \mathcal{B}aire(V)}{l \varepsilon \mathcal{B}aire(V)}$$

Three point-free topologies on the tree List(Nat)

1. frame Loc_{Baire} of fixed points of inductively generated formal topology Baire (only point-free)

2. frame Loc_{funrel} of the pointwise topology on functional relations from Nat to Nat

$$\mathcal{O}_l \equiv \{\ R \varepsilon \text{Funrel}(\text{Nat},\text{Nat})\ \mid\ R \text{ goes through } l\ \}$$

3. frame Loc_{op} of the pointwise topology on operations from Nat to Nat

$$\mathcal{O}_l \equiv \{\ f \in \text{Op}(\text{Nat}, \text{Nat})\ \mid\ f \text{ goes through } l\ \}$$

Topological Benefits of no unique choice

local maps

$$\textit{Loc}_{op} \hookrightarrow \textit{Loc}_{funrel} \hookrightarrow \textit{Loc}_{Baire}$$

BI(Nat) $\Leftrightarrow$ $\textit{Loc}_{funrel} = \textit{Loc}_{Baire}$

AC! $\Rightarrow$ $\textit{Loc}_{op} = \textit{Loc}_{funrel}$

Three formal closed operators on Baire point-free topology

notion of formal closed in [Sambin'03, Sambin'10]

1. J_{Baire} maximum associated interior operator to point-free topology *Loc*$_{Baire}$ (only point-free)

   spread on the tree List(Nat) = inhabited *formal closed* of J_{Baire}

2. J_{funrel} associated to *Loc*$_{funrel}$

   $l \varepsilon\, \mathsf{J}_{funrel}(\mathsf{V}) \equiv \exists\, R$ funrel $\&$ R goes through list l

   $\&$ "finite pieces of R graph" $\subset$ V

3. $\mathtt{J}_{op}$ associated to Loc_{op}

$$l \varepsilon \mathtt{J}_{op}(\mathsf{V}) \equiv \quad \exists f \text{ operation } \& f \text{ goes through list l}$$
$$\& \text{ “finite pieces of f graph“} \subset \mathsf{V}$$

Topological Benefits of no unique choice

$$J_{op} \leq J_{funrel} \leq J_{Baire}$$

AC! $\Leftrightarrow$ $J_{op} = J_{funrel}$

Dependent choice for functional relation $\Leftrightarrow$ $J_{funrel} = J_{Baire}$

on lists $\Leftrightarrow$ any *spread* is inhabited by a functional relation

Dependent choice for operations $\Leftrightarrow$ $J_{op} = J_{Baire}$

on lists $\Leftrightarrow$ any *spread* is inhabited by an operation

Is there an impredicative theory with same benefits as ours??

Coq= Coquand's Calculus of Constructions

= impredicative version of our intensional level

- Is Coq consistent with CT_{λ} + AC ???

  (⇒ it is a proof-as-programs theory)

- Is Coq consistent with CT_{λ} + Bar Induction ???

  (with choice sequences = functional relations)

Is there a constructive IMPREDICATIVE theory satisfying the above properties?

Relevance of Bar Induction

$\mathrm{BI}(\mathcal{T})$= Bar theorem for point-free topology $\mathcal{T}$

-what is the relevance of Bar Induction (not reduced to Fan theorem) in constructive mathematics?

- classically: all countably generated point-free topologies are spatial
from [Fourman-Grayson'82, Valentini'07]
i.e. $\mathrm{BI}(\mathcal{T})$ holds for all formal topologies $\mathcal{T}$ generated from a countable set of axioms

for what $\mathcal{T}$ is extended Bar Induction $\mathrm{BI}(\mathcal{T})$ constructively acceptable?

(beside $\mathcal{T}$=Cantor, Baire, tree point-free topologies)

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