Another Notion of Point in Locale Theory

Yoshihiro Maruyama

Department of Humanistic Informatics, Graduate School of Letters,
Kyoto University, Japan

Workshop on Constructive Aspects of Logic and Mathematics, Kanazawa, March 2010

Outline

Outline

1. Introduction
   - The nature of space: Pointset vs. Pointfree
   - Two notions of point: Prime vs. Maximal
   - Each notion of point induces a duality
2. Another Duality in Locale Theory
   - Non-sobriety of algebraic varieties
   - Duality between co-atomistic frames and T_1 spaces
   - An old problem: What is pointfree T_1 axiom?
3. Application to Pointfree Convex Geometry
   - What is pointfree convex geometry (PCG)?
   - Relationships with domain theory
   - Our method gives a useful duality for PCG

Why Pointfree?

The notion of “point" seems very ideal, since:

- We cannot see any point in the space.
- Point $\fallingdotseq$ Some Ideal, from a duality-theoretic viewpoint. An indeterministic principle is needed to show its exist..

The notion of “region" seems less ideal, since:

- We can see some regions of the space.
- Region $\fallingdotseq$ Formula, from a duality-theoretic viewpoint. Region can be identified through formula.

It seems: the notion of region is epistemologically more certain than that of point; thus region-based geometry is significant.

Why Pointfree?

The notion of “point" seems very ideal, since:

- We cannot see any point in the space.
- Point ≒ Some Ideal, from a duality-theoretic viewpoint. An indeterministic principle is needed to show its exist..

The notion of “region" seems less ideal, since:

- We can see some regions of the space.
- Region ≒ Formula, from a duality-theoretic viewpoint. Region can be identified through formula.

It seems: the notion of region is epistemologically more certain than that of point; thus region-based geometry is significant.

Why Pointfree?

The notion of “point" seems very ideal, since:

- We cannot see any point in the space.
- Point ≒ Some Ideal, from a duality-theoretic viewpoint. An indeterministic principle is needed to show its exist..

The notion of “region" seems less ideal, since:

- We can see some regions of the space.
- Region ≒ Formula, from a duality-theoretic viewpoint. Region can be identified through formula.

It seems: the notion of region is epistemologically more certain than that of point; thus region-based geometry is significant.

The nature of space: Pointset vs. Pointfree

Pointset models:

- General topology: space $=$ topological space $(S, \mathcal{O})$.
  - Algebraic geometry: space $=$ topo. sp. with sheaf.
- Measure theory: space $=$ measurable space (S, Σ).
- Convex geometry: space $=$ convexity space $(S, \mathcal{C})$.

Pointfree (region-based) models:

- Pointfree topology (locale theory, formal topology, etc).
- Pointfree measure theory (σ-complete Bool. alg., etc).
- Pointfree convex geometry (proposed by the speaker).

Duality theory clarifies the relation. b/w pointset and pointfree spaces, or ontological and epistemological aspects of space.

The nature of space: Pointset vs. Pointfree

Pointset models:

- General topology: space $=$ topological space $(S, \mathcal{O})$.
  - Algebraic geometry: space $=$ topo. sp. with sheaf.
- Measure theory: space $=$ measurable space (S, Σ).
- Convex geometry: space $=$ convexity space $(S, \mathcal{C})$.

Pointfree (region-based) models:

- Pointfree topology (locale theory, formal topology, etc).
- Pointfree measure theory (σ-complete Bool. alg., etc).
- Pointfree convex geometry (proposed by the speaker).

Duality theory clarifies the relation. b/w pointset and pointfree spaces, or ontological and epistemological aspects of space.

Two notions of point: Prime vs. Maximal

Point = Prime Object:

- Modern algebraic geometry by Grothendieck et al.
- Traditional locale theory by Isbell, Johnstone et al.

Point = Maximal Object:

- Classical algebraic geometry by Zariski et al.
- Functional analysis by Gelfand et al.

We discuss the latter view in locale theory, since:

- The former view does not work for recovering the points of algebraic varieties from their open set frames.
- Isbell duality b/w spatial frames and sober spaces is based on the former, but algebraic varieties are usually not sober.

Two notions of point: Prime vs. Maximal

Point = Prime Object:

- Modern algebraic geometry by Grothendieck et al.
- Traditional locale theory by Isbell, Johnstone et al.

Point = Maximal Object:

- Classical algebraic geometry by Zariski et al.
- Functional analysis by Gelfand et al.

We discuss the latter view in locale theory, since:

- The former view does not work for recovering the points of algebraic varieties from their open set frames.
- Isbell duality b/w spatial frames and sober spaces is based on the former, but algebraic varieties are usually not sober.

Our main results: 4 dualities

The view "point as maximal" gives us:

- Duality between co-atomistic frames and T_1 spaces
  - giving an answer to an old problem: what is pointfree T_1?
  - N/B: Algebraic varieties are not sober but T_1.

It is useful also in pointfree convex geometry.

- Prime Object = Polytope $\neq$ Point.
- Maximal Object = Point, in most of usual spaces.
- Two dualities induced by the two notions of point.
  - One of them reveals the convexity-theoretic aspect of Hoffman-Mislove-Stralka duality in domain theory based on an observation by K. Keimel.

Our method also gives a duality between measurable spaces and σ-complete Boolean algebras (the details will be omitted).

Our main results: 4 dualities

The view “point as maximal" gives us:

- Duality between co-atomistic frames and T_1 spaces
  - giving an answer to an old problem: what is pointfree T_1?
  - N/B: Algebraic varieties are not sober but T_1.

It is useful also in pointfree convex geometry.

- Prime Object $=$ Polytope $\neq$ Point.
- Maximal Object $=$ Point, in most of usual spaces.
- Two dualities induced by the two notions of point.
  - One of them reveals the convexity-theoretic aspect of Hoffman-Mislove-Stralka duality in domain theory based on an observation by K. Keimel.

Our method also gives a duality between measurable spaces and σ-complete Boolean algebras (the details will be omitted).

Our main results: 4 dualities

The view “point as maximal" gives us:

- Duality between co-atomistic frames and T_1 spaces
  - giving an answer to an old problem: what is pointfree T_1?
  - N/B: Algebraic varieties are not sober but T_1.

It is useful also in pointfree convex geometry.

- Prime Object $=$ Polytope $\neq$ Point.
- Maximal Object $=$ Point, in most of usual spaces.
- Two dualities induced by the two notions of point.
  - One of them reveals the convexity-theoretic aspect of Hoffman-Mislove-Stralka duality in domain theory based on an observation by K. Keimel.

Our method also gives a duality between measurable spaces and σ-complete Boolean algebras (the details will be omitted).

Outline

1. Introduction
   - The nature of space: Pointset vs. Pointfree
   - Two notions of point: Prime vs. Maximal
   - Each notion of point induces a duality
2. Another Duality in Locale Theory
   - Non-sobriety of algebraic varieties
   - Duality between co-atomistic frames and T_1 spaces
   - An old problem: What is pointfree T_1 axiom?
3. Application to Pointfree Convex Geometry
   - What is pointfree convex geometry (PCG)?
   - Relationships with domain theory
   - Our method gives a useful duality for PCG

Locale Theory

Locale theory studies a frame, i.e., a join-complete lattice with finite meets that distribute over arbitrary joins. A frame homo. is a map preserving finite meets and any joins.

- Frame is a pointfree abstraction of topological space.

A proper subset $P(\neq \emptyset)$ of a frame A is a completely prime filter iff P is an upper set closed under finite meets s.t. $\bigvee_{i \in I} a_i \in P$ implies $a_i \in P$ for some $i \in I$.

- A frame A is spatial iff $\forall a, b \in A \, \exists P$ s.t. P separates a, b.
- A point of A means a completely prime filter of A.

A closed set C in a topo. sp. S is irreducible iff if $C = C_1 \cup C_2$ for closed C_1, C_2 then either $C = C_1$ or $C = C_2$.
S is sober iff $\forall$ irreducible closed $C \, \exists! x \in S$ s.t. $C = \mathrm{cl}(\{x\})$.

Locale Theory

Locale theory studies a frame, i.e., a join-complete lattice with finite meets that distribute over arbitrary joins. A frame homo. is a map preserving finite meets and any joins.

- Frame is a pointfree abstraction of topological space.

A proper subset $P(\neq \emptyset)$ of a frame A is a completely prime filter iff P is an upper set closed under finite meets s.t. $\bigvee_{i \in I} a_i \in P$ implies $a_i \in P$ for some $i \in I$.

- A frame A is spatial iff $\forall a, b \in A \, \exists P$ s.t. P separates a, b.
- A point of A means a completely prime filter of A.

A closed set C in a topo. sp. S is irreducible iff if $C = C_1 \cup C_2$ for closed C_1, C_2 then either $C = C_1$ or $C = C_2$.
S is sober iff $\forall$ irreducible closed $C \, \exists ! x \in S$ s.t. $C = \mathrm{cl}(\{x\})$.

Locale Theory

Locale theory studies a frame, i.e., a join-complete lattice with finite meets that distribute over arbitrary joins. A frame homo. is a map preserving finite meets and any joins.

- Frame is a pointfree abstraction of topological space.

A proper subset $P(\neq \emptyset)$ of a frame A is a completely prime filter iff P is an upper set closed under finite meets s.t. $\bigvee_{i\in I} a_i \in P$ implies $a_i \in P$ for some $i \in I$.

- A frame A is spatial iff $\forall a, b \in A\, \exists P$ s.t. P separates a, b.
- A point of A means a completely prime filter of A.

A closed set C in a topo. sp. S is irreducible iff if $C = C_1 \cup C_2$ for closed C_1, C_2 then either $C = C_1$ or $C = C_2$.
S is sober iff $\forall$ irreducible closed $C\, \exists! x \in S$ s.t. $C = \mathrm{cl}(\{x\})$.

Non-sobriety of algebraic varieties

The fundamental relationships b/w frames and topo. spaces:

- a dual adjunction b/w frames and topo. spaces.
- a duality b/w spatial frames and sober topo. spaces.
  - These have been constructivized by [Aczel 2006]. Our results might be constructivized in a similar way.

Most of ordinary topo. spaces $(S, \mathcal{O})$ are sober, i.e.,

- $(S, \mathcal{O})$ can be recovered from the frame $\mathcal{O}$ by taking the space of completely prime filters of $\mathcal{O}$.

But, not all of ordinary topo. spaces are sober.

- Any irreducible alg. variety in $\mathbb{R}^n$ with the Zariski top. is NOT sober if it is not a singleton.
  - e.g., no single point generates a circle, which is irr.

Non-sobriety of algebraic varieties

The fundamental relationships b/w frames and topo. spaces:

- a dual adjunction b/w frames and topo. spaces.
- a duality b/w spatial frames and sober topo. spaces.
  - These have been constructivized by [Aczel 2006]. Our results might be constructivized in a similar way.

Most of ordinary topo. spaces $(S, \mathcal{O})$ are sober, i.e.,

- $(S, \mathcal{O})$ can be recovered from the frame $\mathcal{O}$ by taking the space of completely prime filters of $\mathcal{O}$.

But, not all of ordinary topo. spaces are sober.

- Any irreducible alg. variety in $\mathbb{R}^n$ with the Zariski top. is NOT sober if it is not a singleton.
  - e.g., no single point generates a circle, which is irr..

Non-sobriety of algebraic varieties

The fundamental relationships b/w frames and topo. spaces:

- a dual adjunction b/w frames and topo. spaces.
- a duality b/w spatial frames and sober topo. spaces.
    - These have been constructivized by [Aczel 2006]. Our results might be constructivized in a similar way.

Most of ordinary topo. spaces $(S, \mathcal{O})$ are sober, i.e.,

- $(S, \mathcal{O})$ can be recovered from the frame $\mathcal{O}$ by taking the space of completely prime filters of $\mathcal{O}$.

But, not all of ordinary topo. spaces are sober.

- Any irreducible alg. variety in $\mathbb{R}^n$ with the Zariski top. is NOT sober if it is not a singleton.
    - e.g., no single point generates a circle, which is irr..

How to recover the points of algebraic varieties?

The non-sobriety of alg. varieties means:

- The points of alg. varieties cannot be recovered from their frames by taking the spaces of comp. prime. filters.

It never means:

- They cannot be recovered by ANY means.

Actually, they can be recovered:

- By taking the spaces of maximal join-comp. ideals.
  - An ideal is join-comp. iff it is closed under any joins.

The view "Point as Maximal" is appropriate for alg. varieties.
The view "Point as Prime" does not work for alg. varieties.

How to recover the points of algebraic varieties?

The non-sobriety of alg. varieties means:

- The points of alg. varieties cannot be recovered from their frames by taking the spaces of comp. prime. filters.

It never means:

- They cannot be recovered by ANY means.

Actually, they can be recovered:

- By taking the spaces of maximal join-comp. ideals.
  - An ideal is join-comp. iff it is closed under any joins.

The view “Point as Maximal" is appropriate for alg. varieties.
The view “Point as Prime" does not work for alg. varieties.

The notions of m-spatiality and m-homomorphism

The view “Point as Maximal" induces a duality for T_1 spaces.

Definition (m-spatial)

A frame L is m-spatial iff any distinct $a, b \in L$ can be separated by a max. join-comp. ideal of L.

Frame homo. is NOT the dual of conti. map b/w T_1 spaces.

Definition (m-homo.)

A frame homo. $f : L \to L'$ is an m-homo. iff for any max. join-comp. ideal M of L', $f^{-1}(M)$ is a max. join-comp. ideal of L.

mSpFrm := the cat. of m-spatial frames and m-homos..
$\mathrm{T_1Space}$:= the cat. of T_1 spaces and conti. maps.

The notions of m-spatiality and m-homomorphism

The view “Point as Maximal" induces a duality for T_1 spaces.

Definition (m-spatial)

A frame L is m-spatial iff any distinct $a, b \in L$ can be separated by a max. join-comp. ideal of L.

Frame homo. is NOT the dual of conti. map b/w T_1 spaces.

Definition (m-homo.)

A frame homo. $f : L \to L'$ is an m-homo. iff for any max. join-comp. ideal M of L', $f^{-1}(M)$ is a max. join-comp. ideal of L.

$\mathrm{mSpFrm} :=$ the cat. of m-spatial frames and m-homos..
$\mathrm{T}_1\mathrm{Space} :=$ the cat. of T_1 spaces and conti. maps.

Duality between co-atomistic frames and T_1 spaces

Theorem (the first result)

The categories mSpFrm *and* $\mathrm{T}_1\mathrm{Space}$ *are dually equivalent.*

An algebraic characterization of m-spatiality (a frame is co-atomistic iff any element is the meet of a set of co-atoms):

Proposition

A frame is m-spatial iff it is co-atomistic.

- Isbell duality does not work for recover. points of alg. vars.
  - Alg. vars. are usually non-sober.
- Our duality does work for that, since alg. vars. are T_1.

Duality between co-atomistic frames and T_1 spaces

Theorem (the first result)

The categories mSpFrm *and* $\mathrm{T}_1\mathrm{Space}$ *are dually equivalent.*

An algebraic characterization of m-spatiality (a frame is co-atomistic iff any element is the meet of a set of co-atoms):

Proposition

A frame is m-spatial iff it is co-atomistic.

- Isbell duality does not work for recover. points of alg. vars.
  - Alg. vars. are usually non-sober.
- Our duality does work for that, since alg. vars. are T_1.

Duality between co-atomistic frames and T_1 spaces

Theorem (the first result)

The categories mSpFrm *and* $\mathrm{T}_1\mathrm{Space}$ *are dually equivalent.*

An algebraic characterization of m-spatiality (a frame is co-atomistic iff any element is the meet of a set of co-atoms):

Proposition

A frame is m-spatial iff it is co-atomistic.

- Isbell duality does not work for recover. points of alg. vars.
  - Alg. vars. are usually non-sober.
- Our duality does work for that, since alg. vars. are T_1.

An old problem: What is pointfree T_1 axiom?

This is a problem raised in the 1970s and discussed by [Aczel,Curi 2010], [Rosický et al. 1984], [Fourman 1983], [Dowker et al. 1974], and others.

- [Rosický et al. 1984] (constructively, [Aczel,Curi 2010]): A frame is T_1 iff its prime elements are co-atoms.
  - spatial T_1 frames = sober T_1 spaces.
- But the frames of algebraic varieties are NOT T_1, since
  - the complmnt. of an irr. subvar. is not a co-atom but prime.
- Algebraic varieties are very natural T_1 spaces. Thus this definition seems inadequate.

We propose: T_1 frame = co-atomistic frame, since:

- The frame of any T_1 space is co-atomistic.
- Any co-atomistic frame is the frame of some T_1 space.

An old problem: What is pointfree T_1 axiom?

This is a problem raised in the 1970s and discussed by [Aczel,Curi 2010], [Rosický et al. 1984], [Fourman 1983], [Dowker et al. 1974], and others.

- [Rosický et al. 1984] (constructively, [Aczel,Curi 2010]): A frame is T_1 iff its prime elements are co-atoms.
  - spatial T_1 frames $=$ sober T_1 spaces.
- But the frames of algebraic varieties are NOT T_1, since
  - the complmnt. of an irr. subvar. is not a co-atom but prime.
- Algebraic varieties are very natural T_1 spaces. Thus this definition seems inadequate.

We propose: T_1 frame $=$ co-atomistic frame, since:

- The frame of any T_1 space is co-atomistic.
- Any co-atomistic frame is the frame of some T_1 space.

An old problem: What is pointfree T_1 axiom?

This is a problem raised in the 1970s and discussed by [Aczel,Curi 2010], [Rosický et al. 1984], [Fourman 1983], [Dowker et al. 1974], and others.

- [Rosický et al. 1984] (constructively, [Aczel,Curi 2010]): A frame is T_1 iff its prime elements are co-atoms.
  - spatial T_1 frames $=$ sober T_1 spaces.
- But the frames of algebraic varieties are NOT T_1, since
  - the complmnt. of an irr. subvar. is not a co-atom but prime.
- Algebraic varieties are very natural T_1 spaces. Thus this definition seems inadequate.

We propose: T_1 frame $=$ co-atomistic frame, since:

- The frame of any T_1 space is co-atomistic.
- Any co-atomistic frame is the frame of some T_1 space.

An old problem: What is pointfree T_1 axiom?

This is a problem raised in the 1970s and discussed by [Aczel,Curi 2010], [Rosický et al. 1984], [Fourman 1983], [Dowker et al. 1974], and others.

- [Rosický et al. 1984] (constructively, [Aczel,Curi 2010]): A frame is T_1 iff its prime elements are co-atoms.
  - spatial T_1 frames $=$ sober T_1 spaces.
- But the frames of algebraic varieties are NOT T_1, since
  - the complmnt. of an irr. subvar. is not a co-atom but prime.
- Algebraic varieties are very natural T_1 spaces. Thus this definition seems inadequate.

We propose: T_1 frame $=$ co-atomistic frame, since:

- The frame of any T_1 space is co-atomistic.
- Any co-atomistic frame is the frame of some T_1 space.

Outline

Pointfree Convex Geometry (PCG)

We have proposed Pointfree Convex Geometry. Concepts for PCG are compared with those for locale theory as follows.

	pointfree convex geometry	locale theory
space	convexity space	topological space
region	convex sets	open sets
algebra	convexity algebra	frame
prime point	Scott-open prin. filter	completely prime filter
max. point	maximal prin. filter	max. join-comp. ideal

A convexity space is $(S, \mathcal{C})$ where $\mathcal{C} \subset \mathcal{P}(S)$ such that $\mathcal{C}$ is closed under any intersections and unions of directed sets.

- Reference: “Theory of convex structures" by van de Vel.

Polytopes are the canonical base

Let $(S, \mathcal{C})$, $(S', \mathcal{C}')$ be convexity spaces.

Definition (morphism of convexity spaces)

A map $f : S \to S'$ is a morphism of convexity spaces iff for any $C' \in \mathcal{C}'$, we have $f^{-1}(C') \in \mathcal{C}$.

For $A \subset S$, define $\mathrm{ch}(A) = \bigcap\{C \in \mathcal{C} \,;\, A \subset C\}$.

Definition (polyotpe)

A polytope is defined as the ch of finite points.

- Any $C \in \mathcal{C}$ is the union of a directed set of polytopes.
- The set of polytopes in a convexity space is the canonical base of the space.

Note that there is no such canonical base of a topology.

Polytopes are the canonical base

Let $(S, \mathcal{C})$, $(S', \mathcal{C}')$ be convexity spaces.

Definition (morphism of convexity spaces)

A map $f : S \to S'$ is a morphism of convexity spaces iff for any $C' \in \mathcal{C}'$, we have $f^{-1}(C') \in \mathcal{C}$.

For $A \subset S$, define $\mathrm{ch}(A) = \bigcap\{C \in \mathcal{C}\ ;\ A \subset C\}$.

Definition (polyotpe)

A polytope is defined as the ch of finite points.

- Any $C \in \mathcal{C}$ is the union of a directed set of polytopes.
- The set of polytopes in a convexity space is the canonical base of the space.

Note that there is no such canonical base of a topology.

Polytopes are the canonical base

Let $(S, \mathcal{C})$, $(S', \mathcal{C}')$ be convexity spaces.

Definition (morphism of convexity spaces)

A map $f : S \to S'$ is a morphism of convexity spaces iff for any $C' \in \mathcal{C}'$, we have $f^{-1}(C') \in \mathcal{C}$.

For $A \subset S$, define $\mathrm{ch}(A) = \bigcap\{C \in \mathcal{C} \,;\, A \subset C\}$.

Definition (polyotpe)

A polytope is defined as the ch of finite points.

- Any $C \in \mathcal{C}$ is the union of a directed set of polytopes.
- The set of polytopes in a convexity space is the canonical base of the space.

Note that there is no such canonical base of a topology.

Convexity algebra as pointfree convexity space

Definition (convexity algebra)

A poset L is a convexity algebra iff

- L has arbitrary meets and joins of directed sets;
- joins of directed sets distribute over arbitrary meets.

This is a pointfree abstraction of convexity space.

Let L_1 and L_2 be convexity algebras.

Definition (homomorphism)

A function $f : L_1 \to L_2$ is a homomorphism from L_1 to L_2 iff f preserves arbitrary meets and joins of directed sets.

$\mathrm{CA} :=$ the cat. of conv. algs. and homos..
$\mathrm{CS} :=$ the cat. of conv. spaces and their morphisms.

Convexity algebra as pointfree convexity space

Definition (convexity algebra)

A poset L is a convexity algebra iff

- L has arbitrary meets and joins of directed sets;
- joins of directed sets distribute over arbitrary meets.

This is a pointfree abstraction of convexity space.
Let L_1 and L_2 be convexity algebras.

Definition (homomorphism)

A function $f : L_1 \to L_2$ is a homomorphism from L_1 to L_2 iff f preserves arbitrary meets and joins of directed sets.

$\mathrm{CA} :=$ the cat. of conv. algs. and homos..
$\mathrm{CS} :=$ the cat. of conv. spaces and their morphisms.

The two notions of point for convexity algebras

Let L be a convexity algebra.

Definition (Scott-open principal filter)

$P \subset L$ is a Scott-open principal filter iff P is a principal filter s.t.

- for any directed $\{a_i \in L\,;\, i \in I\}$, $\bigvee_{i \in I} a_i \in P$ implies $a_i \in P$ for some $i \in I$.

A maximal principal filter is defined in the usual way.

- Why principal? B/c principal filters = meet-complete filters.
- These are the two notions of point for convexity algebras.

$\mathrm{Spec}(L) :=$ the conv. sp. of Scott-open prin. filters.
$\mathrm{Spm}(L) :=$ the conv. sp. of maximal prin. filters.

The two notions of point for convexity algebras

Let L be a convexity algebra.

Definition (Scott-open principal filter)

$P \subset L$ is a Scott-open principal filter iff P is a principal filter s.t.

- for any directed $\{a_i \in L \,;\, i \in I\}$, $\bigvee_{i \in I} a_i \in P$ implies $a_i \in P$ for some $i \in I$.

A maximal principal filter is defined in the usual way.

- Why principal? B/c principal filters $=$ meet-complete filters.
- These are the two notions of point for convexity algebras.

$\mathrm{Spec}(L) :=$ the conv. sp. of Scott-open prin. filters.
$\mathrm{Spm}(L) :=$ the conv. sp. of maximal prin. filters.

Adjunction and duality for PCG

$\mathrm{Conv}(S) :=$ the conv. alg. of convex sets in a conv. sp. S.
By defining the arrow parts by taking the inverse images, we get

Theorem (Isbell-type adjunction for PCG)

Spec *and* Conv *form a dual adjunction b/w* CA *and* CS*.*

A conv. sp. is sober iff any polytope is the ch of a unique point.
A conv. alg. is spatial iff any two elements can be separated by a Scott-open prin. filter.

Theorem (Isbell-type duality for PCG)

SpCA *and* SobCS *are dually equivalent via* Spec *and* Conv*.*

Note: spatiality is equivalent to algebraicity.

Adjunction and duality for PCG

$\mathrm{Conv}(S) :=$ the conv. alg. of convex sets in a conv. sp. S.
By defining the arrow parts by taking the inverse images, we get

Theorem (Isbell-type adjunction for PCG)

Spec *and* Conv *form a dual adjunction b/w* CA *and* CS.

A conv. sp. is sober iff any polytope is the ch of a unique point.
A conv. alg. is spatial iff any two elements can be separated by a Scott-open prin. filter.

Theorem (Isbell-type duality for PCG)

SpCA *and* SobCS *are dually equivalent via* Spec *and* Conv.

Note: spatiality is equivalent to algebraicity.

Adjunction and duality for PCG

$\mathrm{Conv}(S)$:= the conv. alg. of convex sets in a conv. sp. S.
By defining the arrow parts by taking the inverse images, we get

Theorem (Isbell-type adjunction for PCG)

Spec *and* Conv *form a dual adjunction b/w* CA *and* CS.

A conv. sp. is sober iff any polytope is the ch of a unique point.
A conv. alg. is spatial iff any two elements can be separated by a Scott-open prin. filter.

Theorem (Isbell-type duality for PCG)

SpCA *and* SobCS *are dually equivalent via* Spec *and* Conv.

Note: spatiality is equivalent to algebraicity.

Adjunction and duality for PCG

$\mathrm{Conv}(S) :=$ the conv. alg. of convex sets in a conv. sp. S.
By defining the arrow parts by taking the inverse images, we get

Theorem (Isbell-type adjunction for PCG)

Spec *and* Conv *form a dual adjunction b/w* CA *and* CS.

A conv. sp. is sober iff any polytope is the ch of a unique point.
A conv. alg. is spatial iff any two elements can be separated by a Scott-open prin. filter.

Theorem (Isbell-type duality for PCG)

SpCA *and* SobCS *are dually equivalent via* Spec *and* Conv.

Note: spatiality is equivalent to algebraicity.

Hoffman-Mislove-Stralka duality

In domain theory ([Hoffman,Mislove,Stralka 1974]):
$\mathrm{AlgContLat} \simeq \vee\mathrm{SLat}_0^{\mathrm{op}}$ (with $\mathrm{ContLat} \sim \vee\mathrm{SLat}_0^{\mathrm{op}}$).

- A poset is a continuous lattice iff it is a convexity algebra.
  - see "Continuous lattices and domains" by Gierz et al..
- Then, $\mathrm{AlgContLat} \simeq \mathrm{SobCS}^{\mathrm{op}}$ at the object level.
- Thus, $\vee\mathrm{SLat}_0 \simeq \mathrm{SobCS}$ at the object level.

Actually we have: $\vee\mathrm{SLat}_0 = \mathit{Polytope}(\mathrm{CS})$.
Note: The polytopes in a conv. sp. form a $\vee$-semilattice with 0.

- Thus our results reveal the convexity-theoretic meaning of Hoffman-Mislove-Stralka duality.
  - This connection was suggested by K. Keimel. Many thanks.

Hoffman-Mislove-Stralka duality

In domain theory ([Hoffman,Mislove,Stralka 1974]):
$\mathrm{AlgContLat} \simeq \vee\mathrm{SLat}_0^{\mathrm{op}}$ (with $\mathrm{ContLat} \sim \vee\mathrm{SLat}_0^{\mathrm{op}}$).

- A poset is a continuous lattice iff it is a convexity algebra.
  - see “Continuous lattices and domains" by Gierz et al..
- Then, $\mathrm{AlgContLat} \simeq \mathrm{SobCS}^{\mathrm{op}}$ at the object level.
- Thus, $\vee\mathrm{SLat}_0 \simeq \mathrm{SobCS}$ at the object level.

Actually we have: $\vee\mathrm{SLat}_0 = \mathit{Polytope}(\mathrm{CS})$.
Note: The polytopes in a conv. sp. form a $\vee$-semilattice with 0.

- Thus our results reveal the convexity-theoretic meaning of Hoffman-Mislove-Stralka duality.
  - This connection was suggested by K. Keimel. Many thanks.

Hoffman-Mislove-Stralka duality

In domain theory ([Hoffman,Mislove,Stralka 1974]):
$\mathrm{AlgContLat} \simeq \vee\mathrm{SLat}_0^{\mathrm{op}}$ (with $\mathrm{ContLat} \sim \vee\mathrm{SLat}_0^{\mathrm{op}}$).

- A poset is a continuous lattice iff it is a convexity algebra.
  - see “Continuous lattices and domains" by Gierz et al..
- Then, $\mathrm{AlgContLat} \simeq \mathrm{SobCS}^{\mathrm{op}}$ at the object level.
- Thus, $\vee\mathrm{SLat}_0 \simeq \mathrm{SobCS}$ at the object level.

Actually we have: $\vee\mathrm{SLat}_0 = \mathit{Polytope}(\mathrm{CS})$.
Note: The polytopes in a conv. sp. form a $\vee$-semilattice with 0.

- Thus our results reveal the convexity-theoretic meaning of Hoffman-Mislove-Stralka duality.
  - This connection was suggested by K. Keimel. Many thanks.

Hoffman-Mislove-Stralka duality

In domain theory ([Hoffman,Mislove,Stralka 1974]):
$\mathrm{AlgContLat} \simeq \vee\mathrm{SLat}_0^{\mathrm{op}}$ (with $\mathrm{ContLat} \sim \vee\mathrm{SLat}_0^{\mathrm{op}}$).

- A poset is a continuous lattice iff it is a convexity algebra.
  - see “Continuous lattices and domains" by Gierz et al..
- Then, $\mathrm{AlgContLat} \simeq \mathrm{SobCS}^{\mathrm{op}}$ at the object level.
- Thus, $\vee\mathrm{SLat}_0 \simeq \mathrm{SobCS}$ at the object level.

Actually we have: $\vee\mathrm{SLat}_0 = \mathit{Polytope}(\mathrm{CS})$.
Note: The polytopes in a conv. sp. form a $\vee$-semilattice with 0.

- Thus our results reveal the convexity-theoretic meaning of Hoffman-Mislove-Stralka duality.
  - This connection was suggested by K. Keimel. Many thanks.

Problem on the duality for PCG

$\mathbb{R}^n$ with the usual convexity is not a sober conv. sp. (since generally a polytope is not the ch of any point).

- Most of ordinary convexity spaces are NOT sober, while most of ordinary topological spaces are sober.
  - A striking difference b/w topology and conv. geom..
- We wish that ordinary convexity spaces were sober.

This problem can be solved by replacing prime point with maximal point as in the case of algebraic varieties.

- $\mathrm{Spm}(L) :=$ the conv. sp. of max. prin. filters.
- Then, $\mathbb{R}^n \simeq \mathrm{Spm} \circ \mathrm{Conv}(\mathbb{R}^n)$.
- In this sense, max. point is more useful than prime point.

Problem on the duality for PCG

$\mathbb{R}^n$ with the usual convexity is not a sober conv. sp.
(since generally a polytope is not the ch of any point).

- Most of ordinary convexity spaces are NOT sober, while most of ordinary topological spaces are sober.
  - A striking difference b/w topology and conv. geom..
- We wish that ordinary convexity spaces were sober.

This problem can be solved by replacing prime point with maximal point as in the case of algebraic varieties.

- $\mathrm{Spm}(L) :=$ the conv. sp. of max. prin. filters.
- Then, $\mathbb{R}^n \simeq \mathrm{Spm} \circ \mathrm{Conv}(\mathbb{R}^n)$.
- In this sense, max. point is more useful than prime point.

Problem on the duality for PCG

$\mathbb{R}^n$ with the usual convexity is not a sober conv. sp.
(since generally a polytope is not the ch of any point).

- Most of ordinary convexity spaces are NOT sober, while most of ordinary topological spaces are sober.
  - A striking difference b/w topology and conv. geom..
- We wish that ordinary convexity spaces were sober.

This problem can be solved by replacing prime point with maximal point as in the case of algebraic varieties.

- $\mathrm{Spm}(L) :=$ the conv. sp. of max. prin. filters.
- Then, $\mathbb{R}^n \simeq \mathrm{Spm} \circ \mathrm{Conv}(\mathbb{R}^n)$.
- In this sense, max. point is more useful than prime point.

m-sobriety and m-spatiality

Definition (m-sober)

A conv. sp. S is m-sober iff for any $x \in S$, $\{x\}$ is convex.

Most of ordinary convexity spaces are m-sober.

Definition (m-spatial)

A convexity algebra is m-spatial iff any two elements can be separated by a max. prin. filter.

mSpCA := the cat. of m-spatial conv. algs. and m-homos.
mSobCS := the cat. of m-sober convexity spaces.

m-sobriety and m-spatiality

Definition (m-sober)

A conv. sp. S is m-sober iff for any $x \in S$, $\{x\}$ is convex.

Most of ordinary convexity spaces are m-sober.

Definition (m-spatial)

A convexity algebra is m-spatial iff any two elements can be separated by a max. prin. filter.

mSpCA := the cat. of m-spatial conv. algs. and m-homos.
mSobCS := the cat. of m-sober convexity spaces.

Another duality for PCG

Theorem (another duality for PCG)

$$\mathrm{Spm} : \mathrm{mSpCA} \simeq \mathrm{mSobCS}^{\mathrm{op}} : \mathrm{Conv}^{\mathrm{op}}.$$

Most of ordinary convexities fall into this duality. In this sense, this duality is more useful than the "prime" duality.

- In PCG the proper notion of point seems max. prin. filter
  - since $S \simeq \mathrm{Spm} \circ \mathrm{Conv}(S)$ in ordinary spaces.
- What Scott. prin. filter represents is not point but polytope
  - since $\mathrm{Spec} \circ \mathrm{Conv}(S)$ is the set of polytopes in S.

Note: The space of polytopes in S is the sobrification of S.

Another duality for PCG

Theorem (another duality for PCG)

$$\mathrm{Spm} : \mathrm{mSpCA} \simeq \mathrm{mSobCS}^{\mathrm{op}} : \mathrm{Conv}^{\mathrm{op}}.$$

Most of ordinary convexities fall into this duality. In this sense, this duality is more useful than the “prime" duality.

- In PCG the proper notion of point seems max. prin. filter
  - since $S \simeq \mathrm{Spm} \circ \mathrm{Conv}(S)$ in ordinary spaces.
- What Scott. prin. filter represents is not point but polytope
  - since $\mathrm{Spec} \circ \mathrm{Conv}(S)$ is the set of polytopes in S.

Note: The space of polytopes in S is the sobrification of S.

Conclusions

Motivated by non-sobriety of algebraic varieties,
we consider max. join-comp. ideal as point. The idea gives:

- $\mathrm{CoAtomFrm} \simeq \mathrm{T_1Space}^{\mathrm{op}}$.
  - This suggests: pointfree T_1 axiom $=$ co-atomisticity.
  - Algebraic varieties fall into our duality, not Isbell duality.

In PCG, the view “Point as Prime" gives:

- $\mathrm{SpCA} \simeq \mathrm{SobCS}^{\mathrm{op}}$ with $\mathrm{ConvAlg} \sim \mathrm{ConvSp}^{\mathrm{op}}$.
  - This reveals the convexity-theoretic meaning of Hoffman-Mislove-Stralka duality in domain theory.

But ordinary conv. spaces do not fall into the above duality.
The view “Point as Maximal" solves this and gives:

- $\mathrm{mSpCA} \simeq \mathrm{mSobCS}^{\mathrm{op}}$.
  - Most of ordinary spaces fall into this duality.
  - Scott. prin. filter $=$ Polytope. Max. prin. filter $\fallingdotseq$ Point.

Conclusions

Motivated by non-sobriety of algebraic varieties, we consider max. join-comp. ideal as point. The idea gives:

- $\mathrm{CoAtomFrm} \simeq \mathrm{T_1Space}^{\mathrm{op}}$.
  - This suggests: pointfree T_1 axiom $=$ co-atomisticity.
  - Algebraic varieties fall into our duality, not Isbell duality.

In PCG, the view "Point as Prime" gives:

- $\mathrm{SpCA} \simeq \mathrm{SobCS}^{\mathrm{op}}$ with $\mathrm{ConvAlg} \sim \mathrm{ConvSp}^{\mathrm{op}}$.
  - This reveals the convexity-theoretic meaning of Hoffman-Mislove-Stralka duality in domain theory.

But ordinary conv. spaces do not fall into the above duality. The view "Point as Maximal" solves this and gives:

- $\mathrm{mSpCA} \simeq \mathrm{mSobCS}^{\mathrm{op}}$.
  - Most of ordinary spaces fall into this duality.
  - Scott. prin. filter $=$ Polytope. Max. prin. filter $\fallingdotseq$ Point.

Conclusions

Motivated by non-sobriety of algebraic varieties,
we consider max. join-comp. ideal as point. The idea gives:

- $\mathrm{CoAtomFrm} \simeq \mathrm{T_1Space}^{\mathrm{op}}$.
  - This suggests: pointfree T_1 axiom $=$ co-atomisticity.
  - Algebraic varieties fall into our duality, not Isbell duality.

In PCG, the view “Point as Prime" gives:

- $\mathrm{SpCA} \simeq \mathrm{SobCS}^{\mathrm{op}}$ with $\mathrm{ConvAlg} \sim \mathrm{ConvSp}^{\mathrm{op}}$.
  - This reveals the convexity-theoretic meaning of Hoffman-Mislove-Stralka duality in domain theory.

But ordinary conv. spaces do not fall into the above duality.
The view “Point as Maximal" solves this and gives:

- $\mathrm{mSpCA} \simeq \mathrm{mSobCS}^{\mathrm{op}}$.
  - Most of ordinary spaces fall into this duality.
  - Scott. prin. filter $=$ Polytope. Max. prin. filter $\fallingdotseq$ Point.

That's All.

Thank you for your attention!!