Proof-theoretic semantics and constructivity

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Aim of the talk

- An examination of Dummett's idea that some inference rules *confer the meanings* of logical constants that they govern.
    - *Justification of logical laws* or *proof-theoretic semantics*.
- An alternative picture of meaning-conferring different from his own.

Dummett’s picture

Dummett's claim
in *The Logical Basis of Metaphysics* (1991)

- Intuitionistic logic (IL) is based on some *self-justifying* inference rules. But classical logic (CL) is not.
  - ✦ A class of basic inference rules of IL *confer the meanings* of logical constants, hence they are self-justifying.
  - ✦ Other basic rules are *justified* in terms of such rules.

I-rules and E-rules

$$\frac{A \quad B}{A \land B}\ (\land I) \qquad \frac{A_0 \land A_1}{A_i}\ (\land E)$$

$$\frac{A_i}{A_0 \lor A_1}\ (\lor I) \qquad \frac{A \lor B \quad \begin{matrix}[A]\\ C\end{matrix} \quad \begin{matrix}[B]\\ C\end{matrix}}{C}\ (\lor E)$$

etc.

Meaning-conferring rules

- First, either I-rules or E-rules are taken to be a “base”, that is, *meaning-conferring*.
- Meaning-rules are regarded as mere stipulations. Hence they are valid by stipulation, that is, *self-justifying*.
- Let us take I-rules as a base here.

Fundamental Assumption

- Non-meaning rules are justified by the *Fundamental Assumption* (FA).
- FA: If we have a valid argument for a complex statement, we can construct a valid argument for it which finishes with an application of one of the introduction rules governing its principal operator.

Fundamental Assumption

$$\begin{array}{c}\mathcal{D}\\ A \wedge B\end{array} \quad \Rightarrow \quad \dfrac{\begin{array}{cc}\mathcal{D}_0 & \mathcal{D}_1\\ A & B\end{array}}{A \wedge B}$$

$$\begin{array}{c}\mathcal{D}\\ A \vee B\end{array} \quad \Rightarrow \quad \dfrac{\begin{array}{c}\mathcal{D}_0\\ A\end{array}}{A \vee B} \quad \textit{or} \quad \dfrac{\begin{array}{c}\mathcal{D}_1\\ B\end{array}}{A \vee B}$$

Fundamental Assumption

$$\vdash A \wedge B \quad \Longleftrightarrow \quad \vdash A \text{ and } \vdash B$$

$$\vdash A \vee B \quad \Longleftrightarrow \quad \vdash A \text{ or } \vdash B$$

- FA: an assumption that I-rule is invertible.
  - The range of *direct grounds* of the complex statement is fixed.
  - I-rules and FA jointly embody *definitional biconditionals*.

Harmony as justification

- Non-meaning rules (E-rules) are justified if *harmony* obtains.
- Harmony: Any consequence that E-rules entitle us to draw from the complex statement is already a consequence of its direct grounds.

Harmony

$$\dfrac{\begin{array}{c}\mathcal{D}\\ A_0 \lor A_1\end{array} \qquad \begin{array}{c}[A_0]\\ \mathcal{D}_0\\ C\end{array} \qquad \begin{array}{c}[A_1]\\ \mathcal{D}_1\\ C\end{array}}{C} \quad \overset{\text{FA}}{\Longrightarrow} \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}'\\ A_i\end{array}}{A_0 \lor A_1} \qquad \begin{array}{c}[A_0]\\ \mathcal{D}_0\\ C\end{array} \qquad \begin{array}{c}[A_1]\\ \mathcal{D}_1\\ C\end{array}}{C}$$

$$\overset{\text{reduction}}{\Longrightarrow} \quad \begin{array}{c}\mathcal{D}'\\ A_i\\ \mathcal{D}_i\\ C\end{array}$$

the conclusion of E-rule can be obtained directly.

FA as a precondition

- Finally, check whether FA holds in the *entire* system.
  - ✦ Justification of E-rules depends on FA.
  - ✦ If FA fails in the system as a whole, justification lose its significance.

CL vs IL

$$\vdash A \vee B \quad \Longrightarrow \quad \vdash A \text{ or } \vdash B$$

- FA for disjunction holds in the full system of IL (disjunction property), but fails in CL.
- IL: self-justifying I-rules + justified E-rules.
- CL: since FA fails, justification doesn't work.

FA as a global feature

- Failure of FA in CL is caused in part by non-conservative formulation of negation.
- FA as a precondition for justification is a global feature of entire systems.

Stability

Harmony

- Harmony: Any consequence that E-rules entitle us to draw from the complex statement is already a consequence of its direct grounds.

$$Cn(A) \subset Cn(DG[A])$$

Stability

$$Cn(DG[A]) \subset Cn(A)$$

- Stability: E-rules enable us to draw all consequences that the direct grounds of the statement entitle us to draw.

Stability

$$\frac{A \vee B \quad \begin{matrix}[A] \\ C\end{matrix} \quad \begin{matrix}[B] \\ C\end{matrix}}{C} \ (\vee E)$$

- We can see from the form of the rule that stability obtains.
- How about other constants?

Stability

$$\dfrac{A \wedge B \qquad \begin{matrix}[A, B] \\ C\end{matrix}}{C}\ (\wedge E_G)$$

- General elimination rules (cf. Schroeder-Heister 1984; Negri 2002, von Plato 2003)
  - Equivalent to standard E-rules.
  - Stability can be seen from the form of the rules.

Harmony and stability

$$Cn(A) = Cn(DG[A])$$

- Harmony: the *inverses* of I-rules (FA) justify E-rules (through operations on deductions). The former is not weaker than the latter.
- Stability: E-rules are not weaker than the inverses of I-rules.
- H&S: A symmetry between I-&E-rules.

Read-off rules

$$\frac{A_0 \wedge A_1}{A_i}(\wedge E) \quad \Rightarrow \quad \begin{array}{l} \vdash A \wedge B \Rightarrow \\ \quad \vdash A \text{ and } \vdash B \end{array}$$

- Consider whether E-rules *justify* the inverses of I-rules.
- Such pairs of I-&E-rules are equivalent to the definitional biconditionals.
- Let us call such pairs *read-off* rules.

Read-off rules

$$\frac{A \vee B \quad \begin{array}{c}[A] \\ C\end{array} \quad \begin{array}{c}[B] \\ C\end{array}}{C}\,(\vee E) \quad \underset{???}{\Rightarrow} \quad \vdash A \vee B \Rightarrow \vdash A \text{ or } \vdash B$$

- Disjunction: E-rule cannot be taken to justify the inverse of I-rules.
- The rules for disjunction are harmonious and stable, but not read-off rules.

Read-off rules

- "Read-off": a stronger symmetry than H&S.
  - Dummett doesn't mention it.

➡ An alternative picture of meaning-conferring emerges if we consider systems consisting solely of read-off rules.

- The picture is essentially what is presented in Sambin, Battilotti and Faggian (2000).

In merely H&S systems

I-rules + E-rules ≠ definitional biconditionals

I-rules + FA = definitional biconditionals

- One-half of definitional biconditionals are realized only as a *global* feature (FA).
- E-rules are only justified. The symmetry between the rules doesn't play any role in meaning-conferring.

In “read-off” systems

I-rules + E-rules = definitional biconditionals

- Definitional biconditionals are *locally* realized.
- The symmetry between the rules has a significance in meaning-conferring.
- FA as a global feature must hold to warrant that the “definitions” realized by rules are correct.

Why harmony and stability?

E-rules as a base

H&S: $Cn(A \vee B) = Cn(DG[A \vee B])$

$$\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}$$

- If we take E-rules (L-rules) as the “base”, then the rules can be seen as read-off rules.

E-rules as a base

definitional biconditional:

$$\Gamma, A \vee B \vdash C \iff \Gamma, A \vdash C \text{ and } \Gamma, B \vdash C$$

$$\dfrac{\Gamma \vdash A_i \qquad \dfrac{A_0 \vee A_1 \vdash A_0 \vee A_1}{A_i \vdash A_0 \vee A_1}}{\Gamma \vdash A_0 \vee A_1}$$

The inverse of L-rule

The inverse of L-rule justifies R-rule.

E-rules as a base

Conversely,

$$(\vee R)\ \dfrac{\dfrac{A_i \vdash A_i}{A_i \vdash A_0 \vee A_1} \quad \Gamma, A_0 \vee A_1 \vdash C}{\Gamma, A_i \vdash C}$$

R-rules justifies the inverse of L-rule.

Therefore L-&R-rules are equivalent to the definitional biconditional.

E-rules as a base

- This feature does not differentiate IL from CL, so has no use for Dummett's purpose.
- It must be always possible to take I-rules as the base.
- The "reading-off" picture is not suitable for him here.

Multiple-conclusion

- How to obtain a pair of read-off rules for disjunction with I-rules as the base?
- A natural way is to allow for *multiple-conclusion* (cf. Read 2000).

$$(\vee I)\ \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}$$

$$(\vee E)\ \frac{\Gamma \vdash A \vee B, \Delta \quad A, \Sigma \vdash \Pi \quad B, \Sigma \vdash \Pi}{\Gamma, \Sigma \vdash \Delta, \Pi}$$

Definitional biconditional: $\Gamma \vdash A \vee B, \Delta \iff \Gamma \vdash A, B, \Delta$

FA

$$\dfrac{\Gamma \vdash A \vee B, \Delta \quad A, \Sigma \vdash \Pi \quad B, \Sigma \vdash \Pi}{\Gamma, \Sigma \vdash \Delta, \Pi}\ (\vee E)$$

$$\dfrac{\dfrac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta} \quad A, \Sigma \vdash \Pi \quad B, \Sigma \vdash \Pi}{\Gamma, \Sigma \vdash \Delta, \Pi}$$

reduction

$$\dfrac{\dfrac{\Gamma \vdash A, B, \Delta \quad A, \Sigma \vdash \Pi}{\Gamma, \Sigma \vdash B, \Delta, \Pi} \quad B, \Sigma \vdash \Pi}{\Gamma, \Sigma \vdash \Delta, \Pi}$$

The inverse of I-rule justifies E-rule

Definitional biconditional: $\Gamma \vdash A \vee B, \Delta \iff \Gamma \vdash A, B, \Delta$

$$\frac{\Gamma \vdash A \vee B, \Delta \quad A, \Sigma \vdash \Pi \quad B, \Sigma \vdash \Pi}{\Gamma, \Sigma \vdash \Delta, \Pi} \; (\vee E)$$

$$\Downarrow \quad \Sigma := \emptyset, \Pi := \{A, B\}$$

$$\frac{\Gamma \vdash A \vee B, \Delta \quad A \vdash A, B \quad B \vdash A, B}{\Gamma \vdash A, B, \Delta}$$

E-rule justifies the inverse of I-rule.

Thus, I-&E-rules are equivalent to the definitional biconditional.

Multiple-conclusion

- Multiple-conclusion rules make possible a "reading-off" formulation of disjunction.
- But they open the way for a formulation of classical logic based on self-justifying (meaning-conferring) rules.
- The "reading-off" picture is not suitable for Dummett's purpose here, too.

Conclusion

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- Two different symmetry between the rules: H&S and reading-off.
- Two different pictures of meaning-conferring by inference rules.
- The "reading-off" picture is not suitable for Dummett's intent.
- But it is surely a viable alternative.

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