A Reverse Mathematics for Feasiblity

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We can use LEM, since all objects are finite!

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   - Base Theory $\mathbf{V}^-$ (corresponding to $\mathbf{RCA}_0$);
   - Δ_1^{PL}-reducibility (corr. to Turing-reducibility).

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- I have refined Cook's results by replacing the base theory with weaker one and continued BRM.

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Cook's base theory $\mathbf{V}^0$ consists of

- axioms of discrete-ordered semi-ring;
- $n \in X \to n < |X|, \; |X| - 1 \in X$;
- Δ_0^B-bounded comprehension(-bCA): $(\exists X \leq t)(\forall x < t)(x \in X \leftrightarrow \varphi(x))$ for Δ_0^B φ.

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  $(\exists X \le t)(\forall x < t)(x \in X \leftrightarrow \varphi(x))$ for Δ_0^B φ.

Therefore, it can be said that

BRM is a research on Δ_0^B-reducibility.

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NL: those solvable by NTM within log-time
$=$ those $\mathbf{AC}^0$-reducible to the reachability problem.

Separation of Complexities

While it is known that ...

- $\mathbf{AC}^0 \subsetneq \mathbf{TC}^0 \subset \mathbf{L} \subset \mathbf{NL} \subset \mathbf{P}$.

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For a complexity class $\mathbf{C}$ $(= \mathbf{P}, \mathbf{NL}, \mathbf{L}, \mathbf{TC}^0)$,

- $\delta_{\mathbf{C}}(x, X, Y)$ formalizes "Y is the computation for a chosen $\mathbf{C}$-complete problem with input x, X".
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Thm(i) A poly-bdd $f : 2^{<\omega} \to 2^{<\omega}$ ($|f(X)| \leq p(|X|)$) is Σ^1_1-def. prov. total in **VC** iff "$n \in f(X)$" is in $\mathbf{C}$; (Thus, BRM can be seen as a generalized theory on provably recursive functions.)

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Analogy between SOA and 2-BA

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The analogy we use is:

SOA	unbdd. set qf.	unbdd. number qf.	Δ^1_0	Σ^1_n
2-BA	bdd. set qf.	bdd. number qf.	Δ^B_0	Σ^B_n

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Thm(i) $\mathbf{ACA}_0 + \Delta^1_0$**-AC** is Π^1_2-cons. $/\mathbf{ACA}_0$;

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where Γ**-AC** and Γ**-REPL** are

Γ**-AC** $(\forall n)(\exists X)\varphi(n,X) \rightarrow (\exists Y)(\forall n)\varphi(n,(Y)_n)$;

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To make the comparison precise, we need

the base theory that corresponds to $\mathbf{RCA}_0$.

3. Introduction of Our Frameworks

Our New Base Theory $\mathbf{V}^-$

We can consider $\mathbf{V}^-$, corresponding to $\mathbf{RCA}_0$, by:

- replacing Δ_0^B**-bCA** in $\mathbf{V}^0$ by Δ_1^{PL}**-bCA**;
- adding Σ_1^{PL}-induction;

where Σ_n^{PL} counts alternations of bdd number qf's

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- dropping $|X| - 1 \in X$,

because Cook's |-| should be Δ_0^B-definable:

$$x = \text{length}(X) \leftrightarrow x-1 \in X \wedge (\forall y < |X|)(y \geq x \to y \notin X),$$

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No "robust" machine model for log-time!.

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Moreover, these complete problems are ordered by

Δ_1^{PL}-many-one-reducibility.

4. Result

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I.e., $\exists B' \subset |G| \times |F|$ w/ $\langle g, f\rangle \in B'$ s.t. for $\langle g', f'\rangle \in B'$, $(\forall g'' \leftarrow_G g')(\exists f'' \leftarrow_F f')(\langle g'', f''\rangle \in B')$ and vice versa.

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As a conclusion:

> Finitary combinatorics is quite different from infinitary combinatorics!

Comparison with Other Works

As before, we have results on Δ_1^{PL}-reducibility:

- comparability of well-ordering is $\mathbf{TC}^0$-complete;
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Conclusions

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