On Paths and Points

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The chief part of this is joint work with Thierry Coquand.

WKL, FAN, and Unique Existence

The variables $u, v, w, \ldots$ stand for elements of $\{0,1\}^*$.
As usual the *length* of $u \in \{0,1\}^n$ is defined as $|u| = n$.

We write $u \leqslant w$ if u is a *restriction* of w: that is,

$$u \leqslant w \Leftrightarrow \exists v \ (uv = w) \ .$$

The least element is the *empty sequence* ().

A *(binary) tree* is an inhabited subset T of $\{0,1\}^*$ that is *closed under restrictions*:

$$u \leqslant w \Rightarrow w \in T \Rightarrow u \in T \ .$$

In particular, every tree contains ().

A tree T is *detachable* if

$$\forall u \, (u \in T \vee u \notin T) \, ,$$

and *infinite* if

$$\forall n \, \exists u \; (|u| = n \wedge u \in T) \; .$$

Usually an *infinite path* in a tree T is an infinite sequence

$$\alpha : \mathbb{N} \to \{0, 1\}^*$$

such that T contains all the *initial segments* of α:

$$\overline{\alpha} n = \alpha \, (0) \, , \ldots , \alpha \, (n - 1) \; .$$

Brouwer's Fan Theorem for detachable bars can be put as:

FAN *Every well-founded detachable tree is finite.*

This is the contrapositive of the *Weak König Lemma*:

WKL *Every infinite detachable tree has an infinite path.*

With intuitionistic logic, WKL implies FAN (Ishihara 2006).

Ishihara (1990): WKL is equivalent to

MIN *Every continuous function on a compact space attains its infimum*

J. Berger, Bridges, Sch. (2003): FAN is equivalent to

MIN! *Every continuous function on a compact space that has at most one minimum attains its infimum*

J. Berger, Ishihara; Schwichtenberg (2005): FAN is equivalent to

WKL! *If an infinite detachable tree has at most one infinite path, then it actually has an infinite path.*

A detachable tree T has *at most one infinite path* if

$$\forall n \, \underline{\forall \alpha, \beta \, \exists m} \, \left[\overline{\alpha}(n+m) \in T \wedge \overline{\beta}(n+m) \in T \Rightarrow \overline{\alpha}n = \overline{\beta}n \right],$$

and *uniformly at most one infinite path* if

$$\forall n \, \underline{\exists m \, \forall \alpha, \beta} \, \left[\overline{\alpha}(n+m) \in T \wedge \overline{\beta}(n+m) \in T \Rightarrow \overline{\alpha}n = \overline{\beta}n \right].$$

MIN!, WKL!, and other equivalents of FAN have the following form:

> *If a problem on a compact space has approximate solutions and at most one solution, then it has an exact solution.*

Yet it was all but clear why FAN occurred in this context.

Sch. (2006): FAN is equivalent to

UAM *If a continuous function on a compact space has at most one minimum, then it has uniformly at most one minimum.*

The implication FAN $\Rightarrow$ UAM sharpens FAN $\Rightarrow$ MIN!,
because UAM $\Rightarrow$ MIN! by a well-known metatheorem.

Ishihara (2007): FAN implies

UAP *If a detachable tree has at most one infinite path, then it has uniformly at most one infinite path.*

Ishihara (2007): If a detachable tree T has uniformly at most one infinite path, then it has a *longest path* α:

$$\forall n\ [\exists u\ (|u| = n \wedge u \in T) \Rightarrow \overline{\alpha}n \in T]\ .$$

If T is infinite, then every longest path α is an infinite path in T.

In particular, UAP implies WKL!. In all, UAP is equivalent to FAN.

Unique Paths as Formal Points

We now give a direct and elementary proof of the statement

> *Every infinite tree with uniformly at most one infinite path actually has an infinite path.*

Note that we do not need to suppose that a tree be detachable.

A tree T is a *spread* if every element of T has a successor in T:

$$\forall u \ (u \in T \Rightarrow u0 \in T \vee u1 \in T) \ .$$

Every spread is an infinite tree.

A subset T of $\{0,1\}^*$ is a *chain* if it is *linear* with respect to $\leqslant$:

$$u, u' \in T \Rightarrow |u| \leqslant |u'| \Rightarrow u \leqslant u' \,.$$

Every chain T has at most one element u of any given length:

$$u, u' \in T \Rightarrow |u| = |u'| \Rightarrow u = u' \,;$$

this is equivalent to "T is a chain" if T is closed under restrictions.

The linear spreads are precisely the infinite chains.

A linear spread has exactly one element of any given length.

Let T be a tree. We consider the predicate

$$P(u,k) \equiv \forall w\,(|w| = |u| + k \Rightarrow w \in T \Rightarrow u \leqslant w)\ .$$

Note that $P(u,k)$ implies $P(u,\ell)$ whenever $k \leqslant \ell$. We further set

$$S = \{u \in \{0,1\}^* : \exists k\, P(u,k)\}\,.$$

If T is detachable, then P is decidable, and S is simply existential.

Under suitable conditions this S will be the only infinite path in T.

Lemma 1

(i) S is a tree. If T is infinite, then (ii) S is a chain and (iii) $S \subseteq T$.

Proof. (ii) Let $u, u' \in S$ have length n. Pick k and k' with $P(u,k)$ and $P(u',k')$. We may assume that $k \geqslant k'$. Since T is infinite, there is $v \in T$ such that $|v| = n + k$. Now $u \leqslant v$ and $u' \leqslant v$ by our choice of k, and thus $u = u'$. ■

By an *infinite path* R in a tree T we understand a linear spread (or, equivalently, an infinite chain—see above) such that $R \subseteq T$.

This is nothing but a formal point of the formal topology on $\{0, 1\}^*$, and it is closer in spirit to Brouwer's notion of a choice sequence.

The spreads are precisely the reduced (or formal closed) subsets of $\{0, 1\}^*$ as a formal topology (Maietti 2010, Maietti-Sambin 201?)

Recall that usually an infinite path in a tree T is an infinite sequence

$$\alpha : \mathbb{N} \to \{0,1\}^*$$

such that T contains all the initial segments of α:

$$\overline{\alpha}n = \alpha(0), \ldots, \alpha(n-1) \ .$$

Every α of this sort gives rise to an infinite path in our sense:

$$R_\alpha = \{\overline{\alpha}n : n \geqslant 0\} .$$

Conversely, can one extract an infinite sequence α_R from an infinite path R? If R is detachable, then one can define α_R recursively:

$$\alpha_R(n) = \min\{i \in \{0,1\} : (\overline{\alpha}n)i \in R\} .$$

In general, however, one might need a suitable form of unique countable choice to pick for every n the unique $u \in R$ with $|u| = n$.

A tree T has *uniformly at most one infinite path* (Ishihara 2007) iff

$$\forall n \, \exists m \, \forall v, w \; (|v| = n + m \wedge |w| = n + m \Rightarrow v \in T \wedge w \in T \Rightarrow \overline{v}n = \overline{w}n) \; .$$

For each n, the m as above can be taken as large as one pleases.

Note that in this definition there is no talk of infinite sequences.

Lemma 2 *If T is infinite and has uniformly at most one infinite path, then S is a spread.*

The condition "T is infinite" can be replaced by "T is detachable". In particular, S is a generalisation of Ishihara's longest path.

Proof. By virtue of Lemma 1 it remains to be seen that if $u \in S$, then $u0 \in S$ or $u1 \in S$. Assume that $u \in S$, and take $k \geqslant 1$ with $P(u,k)$. For this k and $n = |u|+1$, take $m \geqslant k$ as in "T has uniformly at most one infinite path". Note that

$$|u| < |u| + k < n + m\,.$$

Since T is infinite, there is $v \in T$ such that $|v| = n + m$. We have $\overline{v}(|u| + k) \in T$ and thus $u \leqslant \overline{v}(|u| + k)$ by our choice of k. Hence $ui \leqslant v$ for some $i \in \{0, 1\}$, for which $P(ui, m)$, and thus $ui \in S$. In fact, if $w \in T$ such that $|w| = n + m$, then $\overline{w}n = \overline{v}n = ui$ by our choice of m. ∎

Theorem 3 *The following conditions are equivalent:*
— T *is infinite and has uniformly at most one infinite path.*
— S *is an infinite path in* T*.*

Proof. By Lemma 1 and Lemma 2 the first item implies the second. As for the converse, assume that S is an infinite path in T. Clearly, T is infinite. To show that T has uniformly at most one infinite path, let n be given. Take any $u \in S$ with $|u| = n$, for which there is m with $P(u, m)$. Any such m is as required: if $v, w \in T$ have both length $n + m$, then $u \leqslant v$ and $u \leqslant w$ by our choice of m; whence $\overline{v}n = u = \overline{w}n$. ∎

We say that $R \subseteq \{0,1\}^*$ is *uniformly absorbing* with respect to T if

$$\forall n\, \exists m\, \forall w\ (|w| = n + m \Rightarrow w \in T \Rightarrow \overline{w}n \in R)\ .$$

Any two uniformly absorbing infinite paths in T are equal.

Corollary 4 *The following conditions are equivalent:*
— T *is infinite and has uniformly at most one infinite path.*
— *There is an infinite path in* T *which is uniformly absorbing.*

Under the equivalent conditions from Theorem 3 and Corollary 4, S is the one and only infinite path in T, and uniformly absorbing.

Appendix: The Unique Solution Property

Bishop-style constructive mathematics with(out) choice.
(Completions without sequences; real numbers: Dedekind cuts.)

Continuity: uniform continuity on compact domains.
Compactness: total boundedness plus completeness.

Setting

Let S be a metric space and $F : S \to \mathbb{R}$ a continuous function.

If S is totally bounded and F uniformly continuous, then $\inf F$ can be computed, in which case we may tacitly assume that $\inf F = 0$.

If S is compact, can one locate a minimum of F? That is, find a point of S at which F attains its infimum?

Heuristics: constructive solutions are continuous in the parameters. To rule out any potential discontinuity, (uniform) uniqueness helps.

Variants of Uniqueness

Suppose that $\inf F = 0$. We agree that $y, y' \in S$ and $\varepsilon, \delta > 0$.

Any such F has *uniformly at most one minimum* if

$$\forall\delta\,\exists\varepsilon\,\forall y, y' \left[F(y) < \varepsilon \wedge F(y') < \varepsilon \Rightarrow d(y, y') < \delta\right]$$

or, equivalently,

$$\forall\delta\,\underline{\exists\varepsilon\,\forall y, y'} \left[d(y, y') \geqslant \delta \Rightarrow F(y) \geqslant \varepsilon \vee F(y') \geqslant \varepsilon\right].$$

In this case F has *at most one minimum*: i.e.,

$$\forall\delta\,\underline{\forall y, y'\,\exists\varepsilon} \left[d(y, y') \geqslant \delta \Rightarrow F(y) \geqslant \varepsilon \vee F(y') \geqslant \varepsilon\right]$$

or more simply but equivalently

$$\forall y, y' \left[d(y, y') > 0 \Rightarrow F(y) > 0 \vee F(y') > 0\right].$$

The Metatheorem of Unique Existence

Let S be complete, and F uniformly continuous.
If $\inf F = 0$ *and F has uniformly at most one minimum,*
then there is $y \in S$ with $F(y) = 0$.

As a proof paradigm this has a considerable history:

Lifshitz 1971, Gelfond 1972, Kreinovich 1979, Bridges 1980, Aczel 1987, Ko 1986, Kohlenbach 1993, Weihrauch 2000, Oliva 2002, Kohlenbach-Oliva 2003, Bauer-Taylor 2005, Brattka 2008, . . .

(to mention for each author only the first printed occurrence)

"Logicians, is there a meta-theorem to explain it?" (Beeson 1985)

The essence of the metatheorem

— can be traced back to Russian recursive mathematics;

— has proved productive in constructive/computable analysis;

— stood right at the beginnings of the so-called proof mining.

The uniqueness hypothesis helps to locate the minimum above any "pure existence proof" tied together with a fragment of the Law of Excluded Middle (e.g. by Bolzano-Weierstraß or WKL).

A parametrised version of the metatheorem subsumes the Implicit Functions Theorem (Diener-Sch. 2009). Before any talk of existence, (uniform) uniqueness with parameters implies (uniform) continuity of the solution as a partial function in the parameters.

Folklore Proof of the Metatheorem

Since $\inf F = 0$ one can choose (!) a sequence (y_n) in S with

$$F(y_n) < 1/n\,,$$

which is Cauchy for F has uniformly at most one minimum.
Since S is complete, (y_n) has a limit y in S, for which $F(y) = 0$.

One only needs S to be complete, and F sequentially continuous.
Uniform uniqueness, however, is essential to get a *Cauchy* sequence.

Even if S fails to be complete, the given data are converted—by countable choice—into an element of the completion of S.

The problem thus provides us, in a sense, with its own solution.
This gave us the clue of how to get by without choice (Sch. 2010).

Doing Without Countable Choice

The completion of S now is the set $\widehat{S}$ of locations (Richman 2000).

Similar methods to define completions without sequences:
Mulvey 1979, Burden and Mulvey 1979, Stolzenberg 1988,
Vickers 2005, Fox 2005, Palmgren 2007, . . .

Let $\mathbb{R}$ denote the set of Dedekind reals: that is, located cuts in $\mathbb{Q}$.

A *location* on S is a function $f : S \to \mathbb{R}$ with $\inf f = 0$ and

$$|f(y) - f(z)| \leqslant d(y,z) \leqslant f(y) + f(z) .$$

The set $\widehat{S}$ of all locations on S is a metric space with metric

$$d(f,g) = \sup |f - g| = \inf (f + g) .$$

There is the isometric embedding

$$S \hookrightarrow \widehat{S}\,,\ \ z \mapsto \widehat{z} = d\,(z,\,\cdot\,)\ ,$$

along which (each point of) S is identified with its image in $\widehat{S}$.

As usual, S is dense in $\widehat{S}$, and S is *complete* if S equals $\widehat{S}$: that is, for every $f \in \widehat{S}$ there is $z \in S$ with $f = \widehat{z}$.

Needless to say, $\widehat{S}$ is complete; and so is $\mathbb{R}$ for $\mathbb{R} \cong \widehat{\mathbb{Q}}$.

Every location measures the distance between itself and the points:

$$d\,(f, \widehat{z}) = f\,(z)\ .$$

Even more clearly, the problem provides us with its own solution!
But why does uniform uniqueness help at all to find the solution?

An Equivalent of Completeness

Definition A metric space S has the *unique solution property* if for every uniformly continuous $F : S \to \mathbb{R}$ with $\inf F = 0$ which has uniformly at most one minimum there is $y \in S$ with $F(y) = 0$.

The metatheorem thus says that every complete metric space has the unique solution property. The converse, however, is also valid:

Theorem 5 *A metric space has the unique solution property if* and only if *it is complete.*

In fact, S is complete already if every location f on S attains its infimum 0 at a point y of S, for which $f = \widehat{y}$ because $d(f, \widehat{y}) = f(y)$.

Alternative proof, for completions with Cauchy sequences:

If (y_n) is a Cauchy sequence in S, then

$$f(y) = \lim_{n\to\infty} d(y, y_n)$$

defines a location f on S such that

$$f(y) = 0 \Leftrightarrow \lim_{n\to\infty} y_n = y\,.$$

Moduli of convergence and of uniqueness correspond to each other.

"Cauchy sequence" and uniform uniqueness have the same form:

$$\forall\delta\,\exists N\,\forall k, k' \,\Big[k \geqslant N \wedge k' \geqslant N \,\Rightarrow\, d(y_k, y_{k'}) < \delta\Big]$$

$$\forall\delta\,\exists\varepsilon\,\forall y, y' \,\Big[F(y) < \varepsilon \wedge F(y') < \varepsilon \,\Rightarrow\, d(y, y') < \delta\Big]$$

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