Domain Representations Derived from Dyadic Subbases

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Kyoto University

Workshop on Constructive Aspects of Logic and Mathematics
March 11, 2010, Kanazawa

Computation over Topological Spaces

- (Admissible) representation in 2^ω or $\mathbb{N}^\omega$.
  $\cdots$ quotient of the digital code space. [TTE, QCB, ...]

- Embedding in $\mathbb{T}^\omega$ for $\mathbb{T} = \{0, 1, \perp\}$.
  $\cdots$ subspace of a code space with bottoms [T].

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Embedding in $\mathbb{T}^\omega$ for $\mathbb{T} = \{0, 1, \perp\}$.
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This work started with

- Gray-embedding of the unit interval [0,1]
- IM2-machines which are working on (subsets of) $\mathbb{T}^\omega$.

As generalization, we studied

- Dyadic subbase,
- Domain representation as minimal limit sets,
- Uniform domain and uniform space.

In this talk, we connect them. that is,

- Construct a domain representation from a dyadic subbase.
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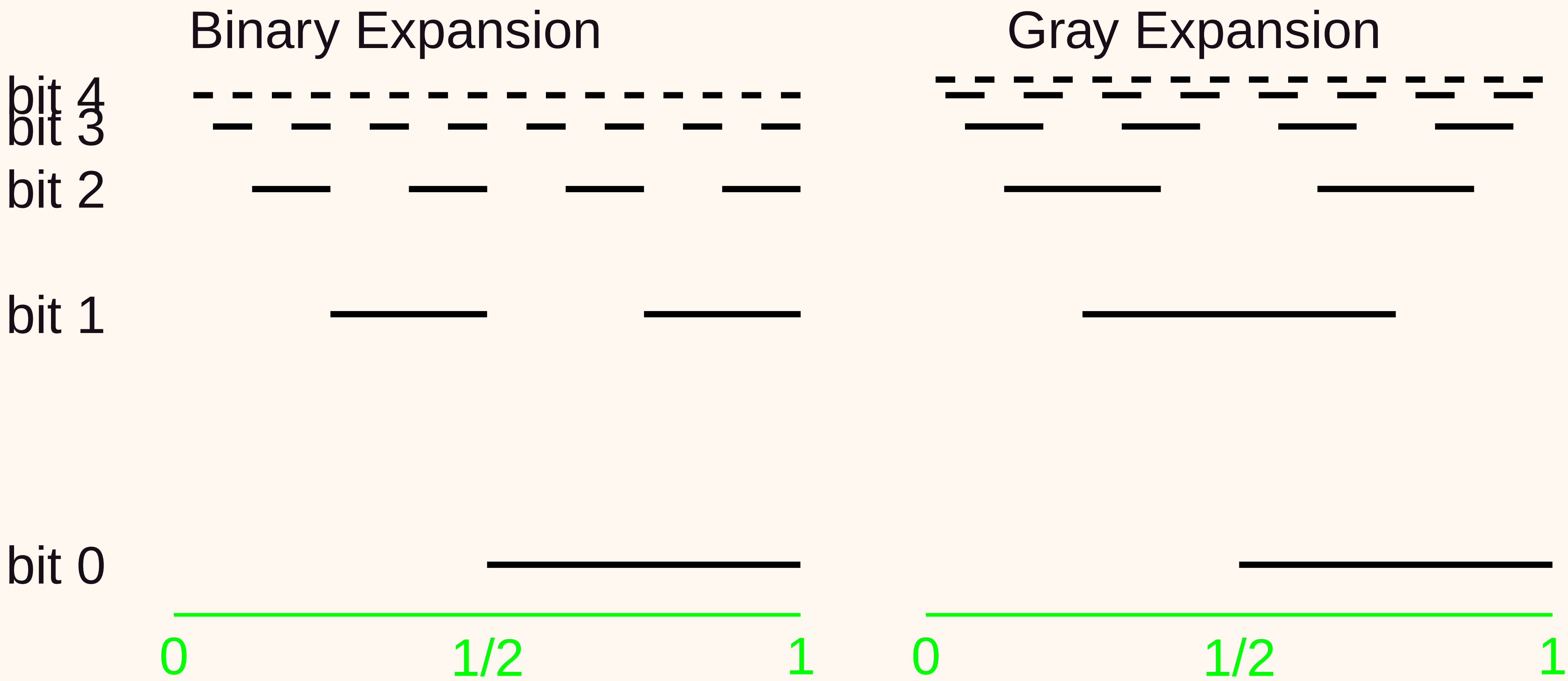

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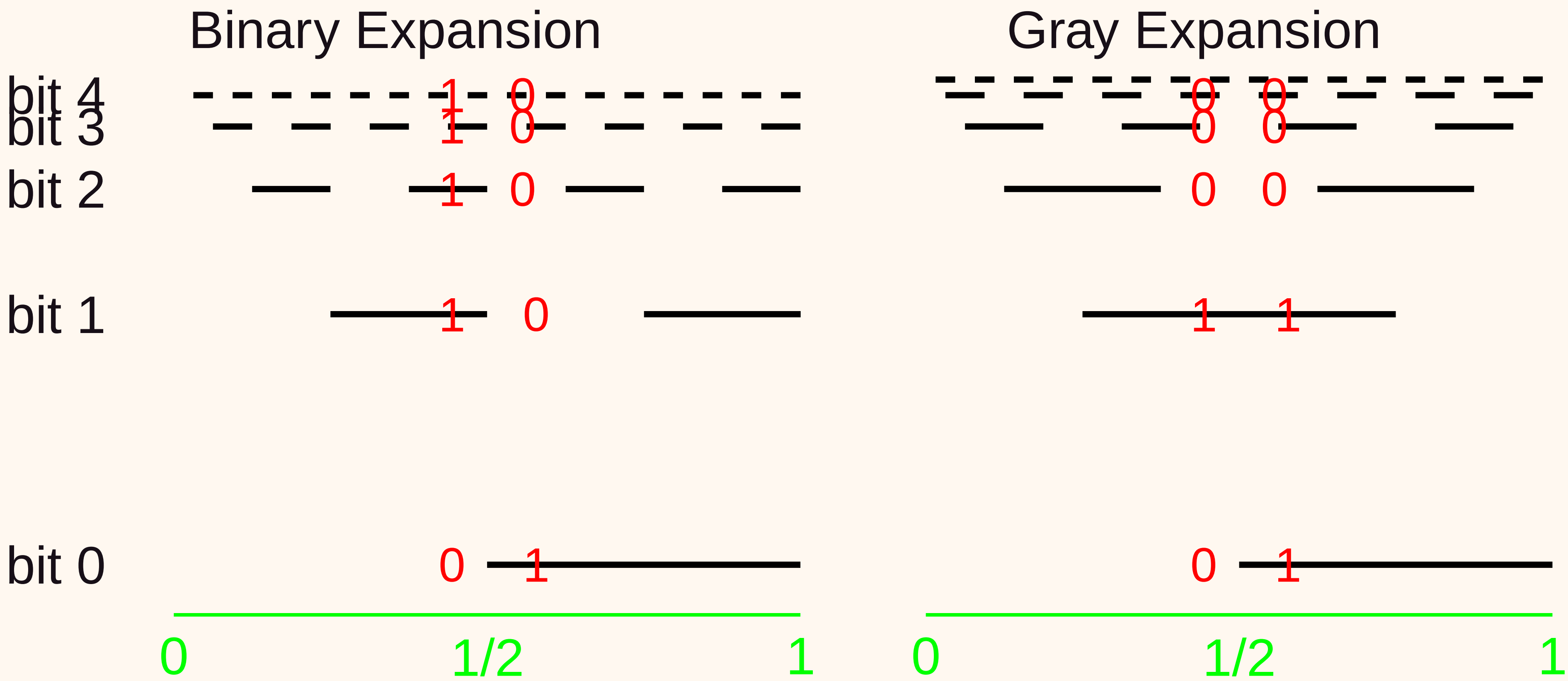

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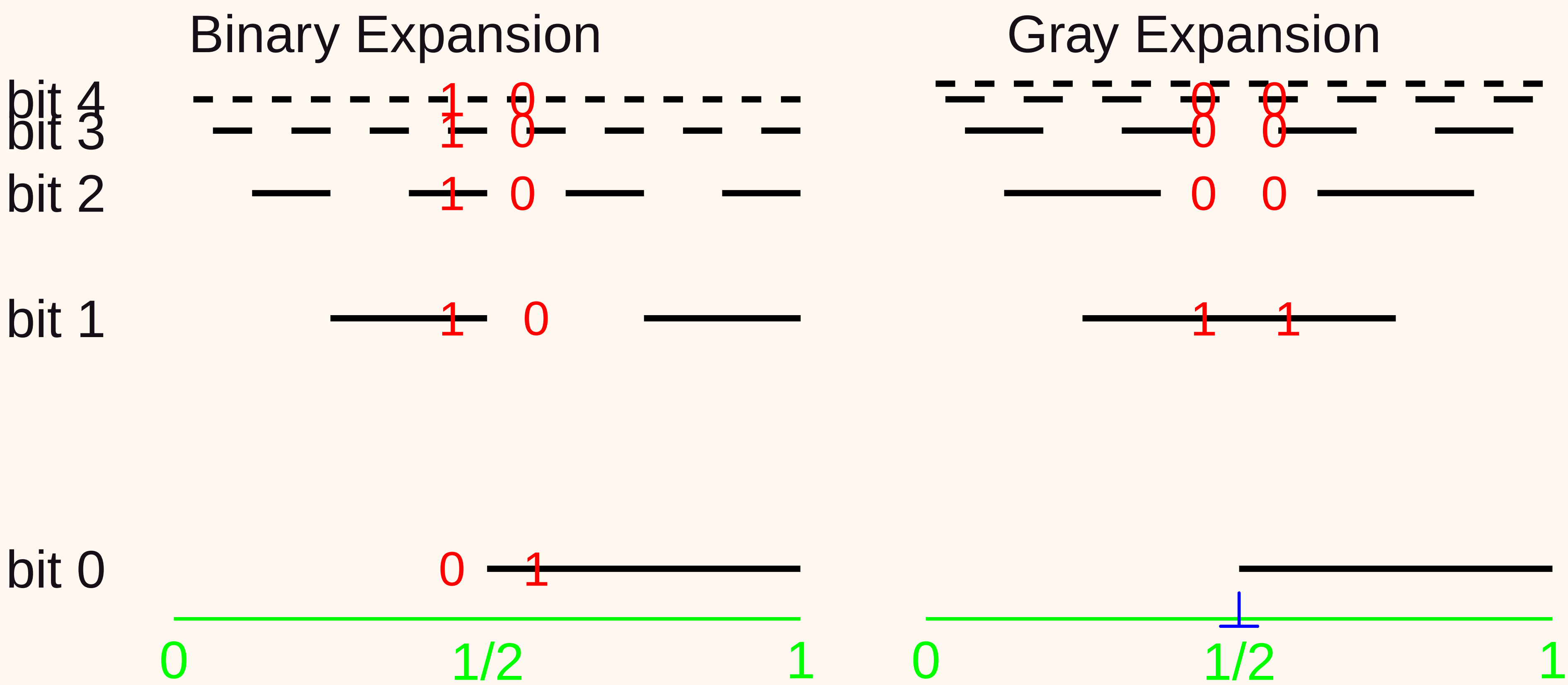

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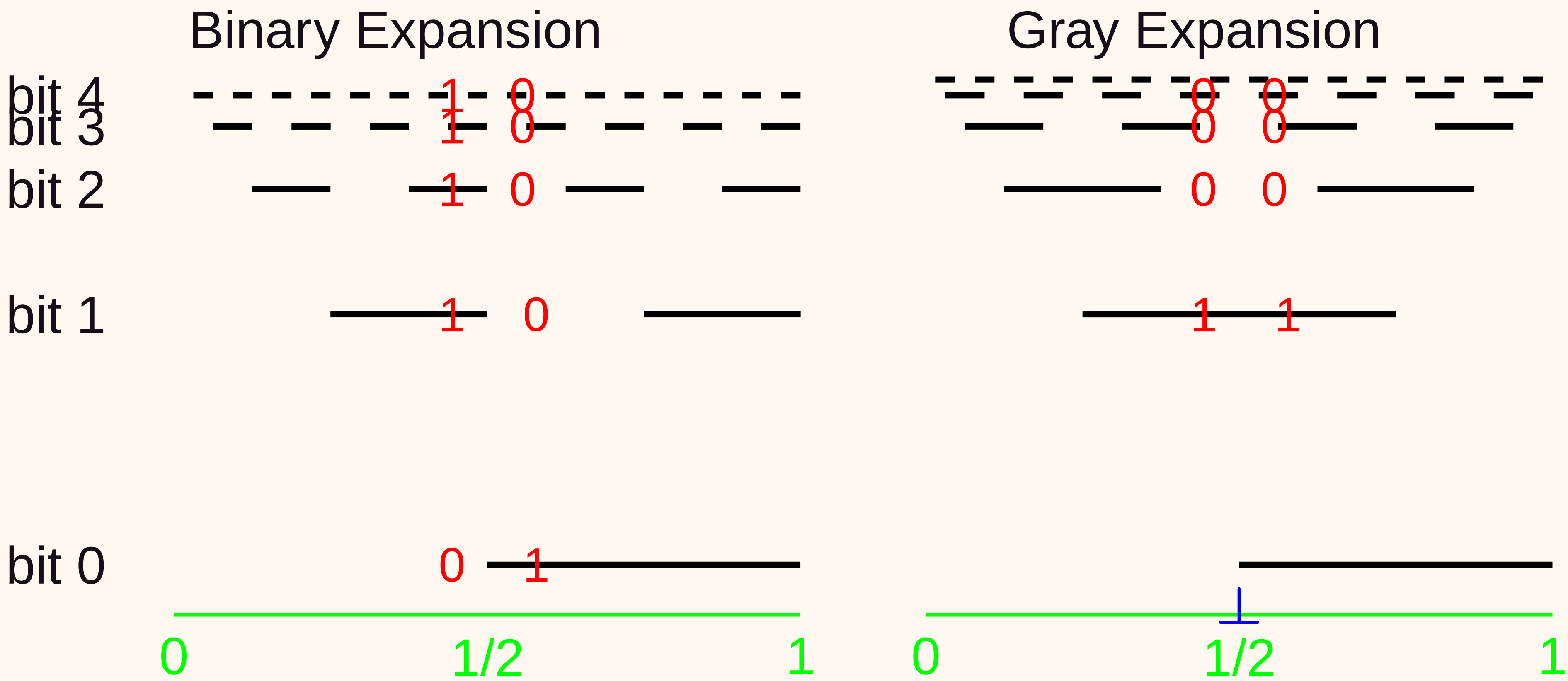

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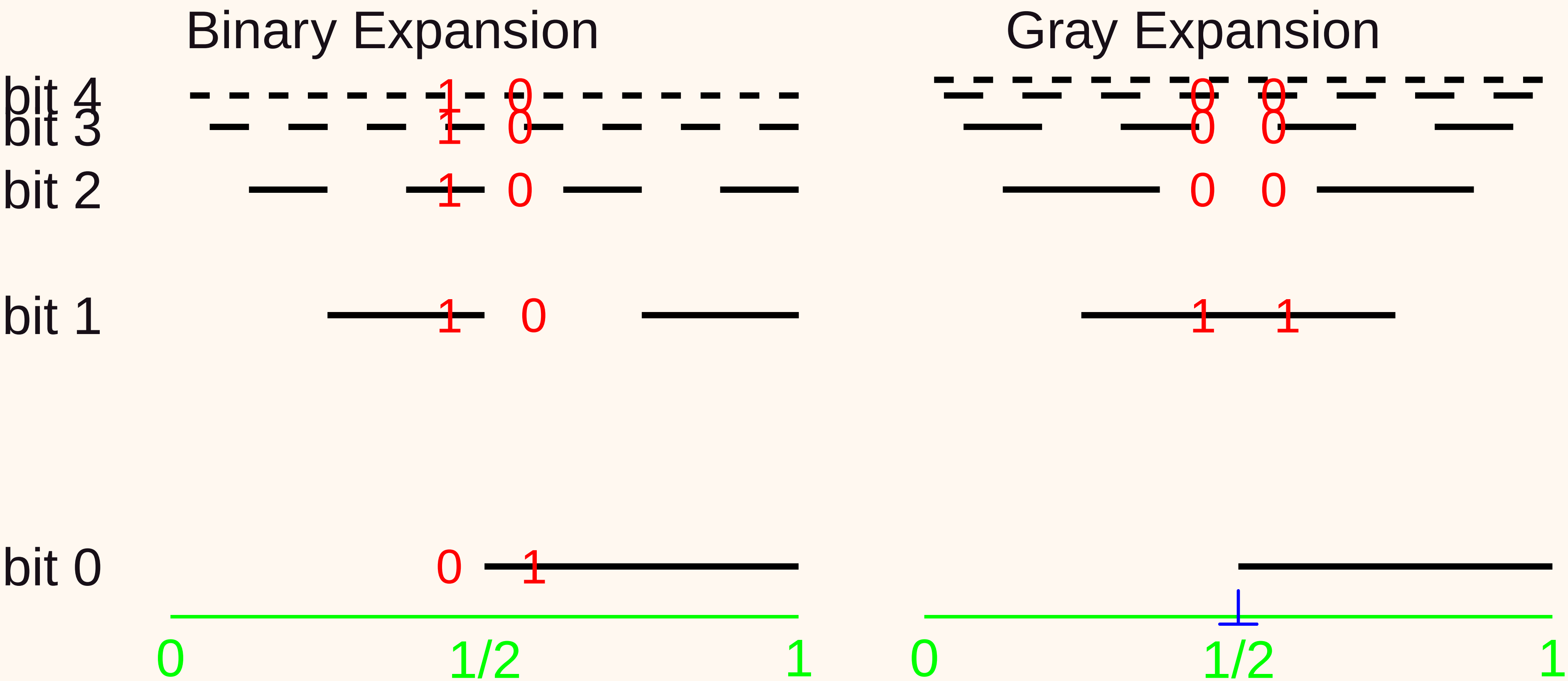
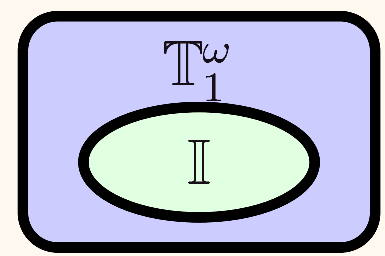


- $\varphi_G(1/2) = \perp 1000\ldots$.
- φ_G: topological embedding of $\mathbb{I}$ in $\mathbb{T}_1^\omega$, which is a subset of $\mathbb{T}^\omega$ at most one $\perp$ exists in each sequence. ([T],[Gianantonio])



- Topology of $\mathbb{T}^\omega$: product topology, (= the Scott Topology on $(\mathbb{T}^\omega, \leq)$).

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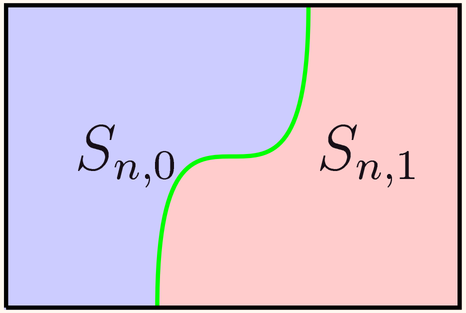

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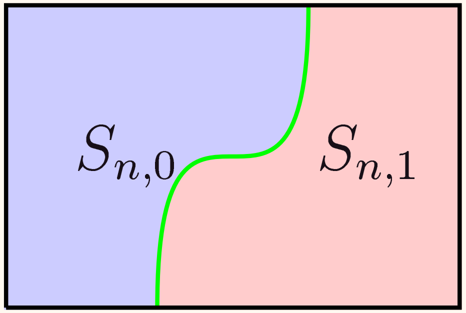

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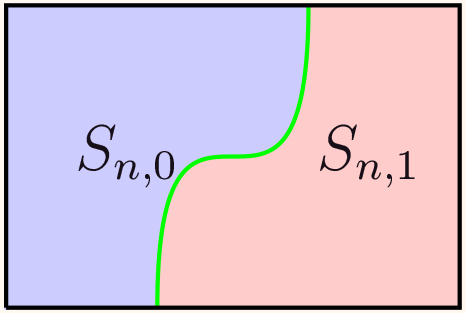
  - regular open = interior of closure is itself.
  - $S_{n,\perp} = \{x \in X \mid \varphi(x)(n) = \perp\}$ is nowhere dense. (it does not contain an open set).
  - The fact $\varphi(x)(n) = \perp$ is not computable. (open set as finitely observable property.) $\perp$ = uncomputable.

Which kind of embeddings in $\mathbb{T}^\omega$? (cont.)

- If $\varphi(x)(n) = \bot$, then x is on the boundary of 0 and 1.
- $\mathrm{cl}\, S_{n,0} = S_{n,0} \cup S_{n,\bot}$,
  $\mathrm{cl}\, S_{n,1} = S_{n,1} \cup S_{n,\bot}$.
- Through this kind of embedding in $\mathbb{T}^\omega$ (with a condition), we can talk about the boundary of basic open sets which are important ex. for dimension theory.
- It is related to domain representation and computation! (as we will see.)



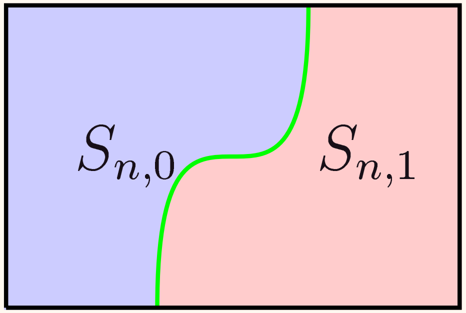
Dyadic Subbase

On the other hand, from a subbase $S = \{S_{n,i} \mid n < \omega, i < 2\}$ which satisfies property $(**)$, we can define embedding $\varphi_S : X \to \mathbb{T}^\omega$ as

$$\varphi_S(x)(n) = \begin{cases} 0 & (x \in S_{n,0}) \\ 1 & (x \in S_{n,1}) \\ \bot & (\text{otherwise}) \end{cases}$$

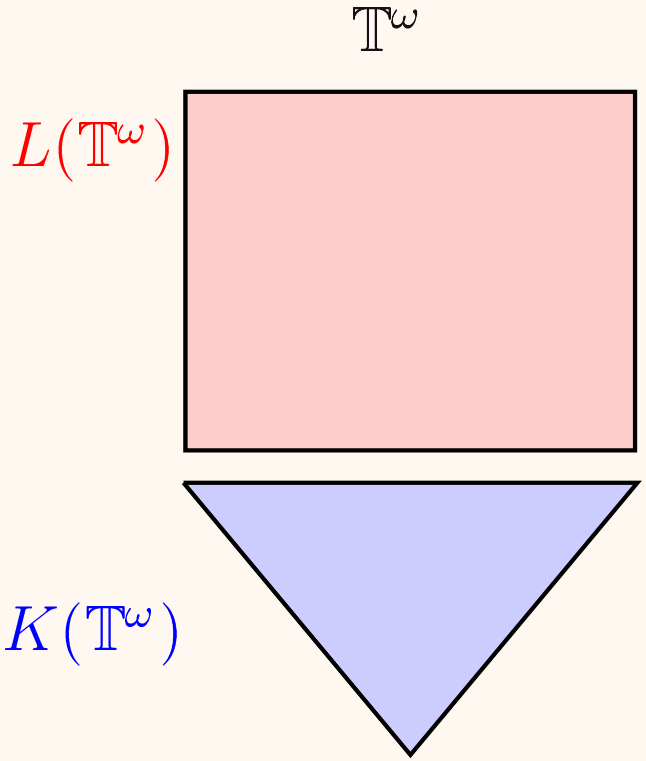
Definition 1 *$S = \{S_{n,i} \mid n < \omega, i < 2\}$ is a dyadic subbase of X if*

1. *S forms a subbase,*
2. *$S_{n,i}$: regular open.*
3. *$S_{n,1} = \mathrm{ext}\ S_{n,0}$ (thus, $S_{n,0} = \mathrm{ext}\ S_{n,1}$).*



Order Structure of $\mathbb{T}^\omega$

- For $p \in \mathbb{T}^\omega$, we call each appearance of 0 or 1 in p a digit of p.
- $K(\mathbb{T}^\omega) = \{p \in \mathbb{T}^\omega : p$ has finite number of digits.$\}$.
- $\mathbb{T}^\omega$ forms an ω-algebraic domain with $K(\mathbb{T}^\omega)$ the set of compact elements.
- $L(\mathbb{T}^\omega) = \mathbb{T} \setminus K(\mathbb{T}^\omega)$.
- $p \leq q$ if $p(n) = c$ implies $q(n) = c$ for $c = 0, 1$.
- $\mathrm{dom}(p) = \{n : p(n) \neq \bot\}$.



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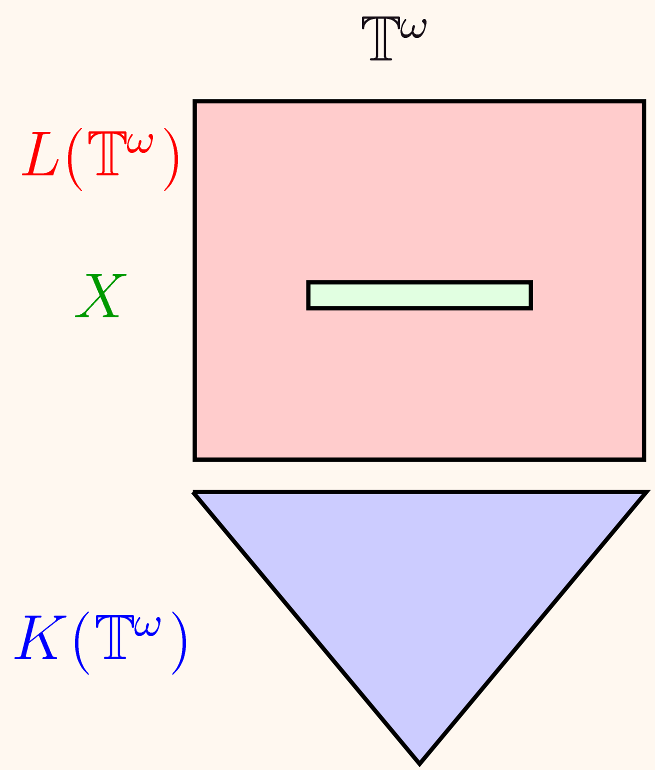
Proposition For a dyadic subbas S of a Hausdorff space X, $\varphi_S(X) \subset L(\mathbb{T}^\omega)$.



$S(d)$ and $\bar{S}(d)$ in the domain $\mathbb{T}^\omega$

For a dyadic subbase S of X and $d \in K(\mathbb{T}^\omega)$, define

$$S(d) = \bigcap_{n \in \mathrm{dom}(d)} S_{n,d(n)}$$

$$\bar{S}(d) = \bigcap_{n \in \mathrm{dom}(d)} \mathrm{cl}\, S_{n,d(n)} = \bigcap_{n \in \mathrm{dom}(d)} (S_{n,d(n)} \cup S_{n,\perp})$$

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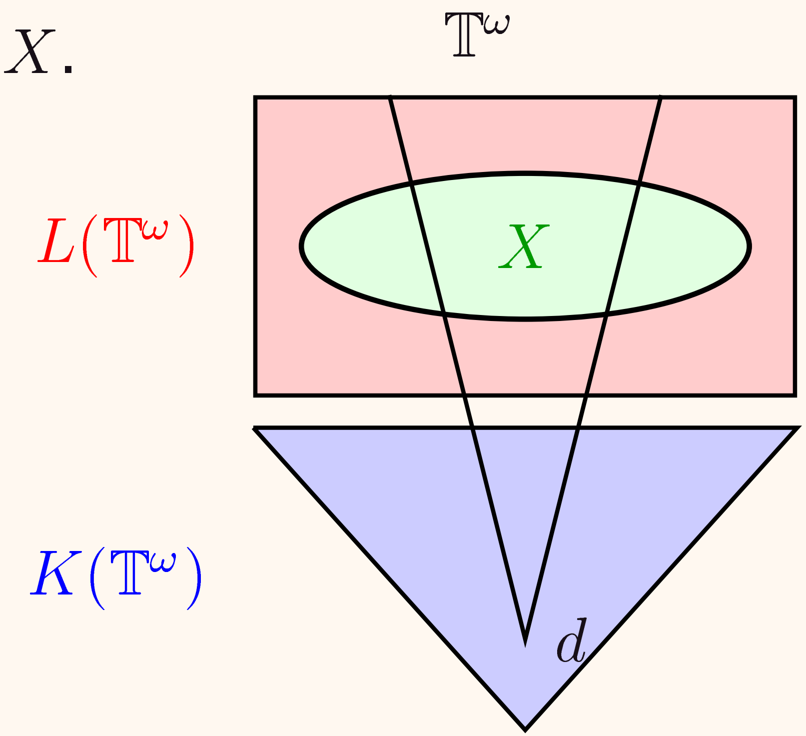

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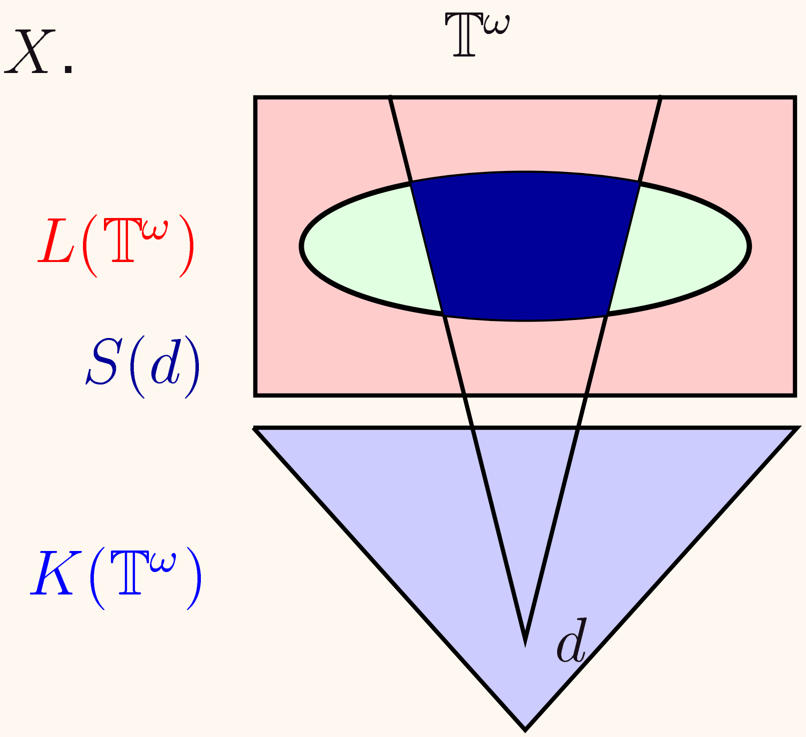

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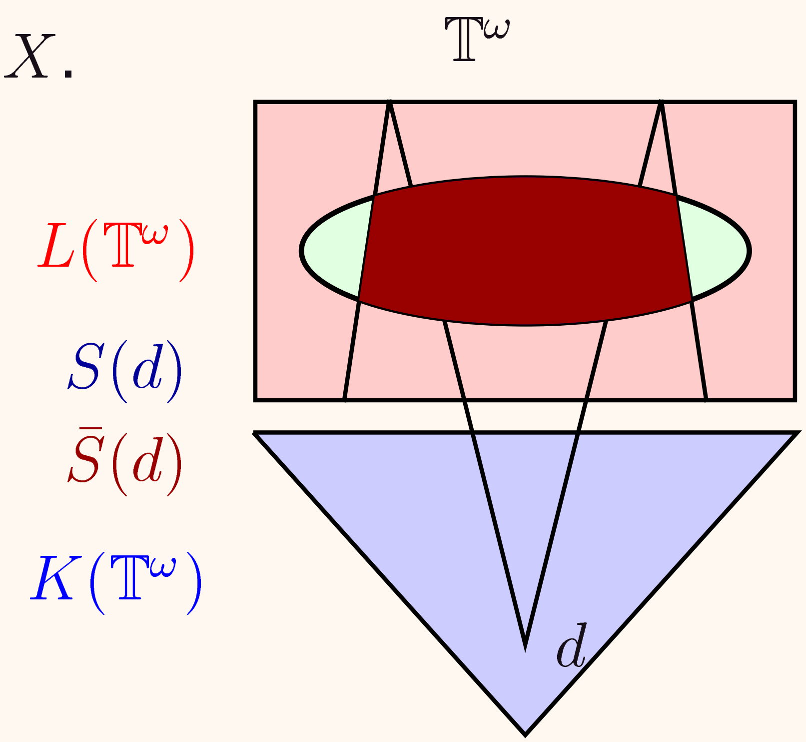
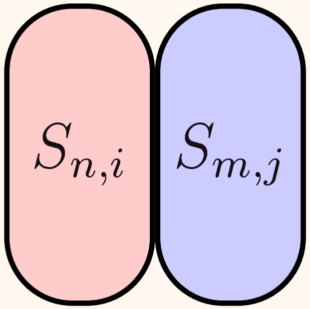


Proper dyadic subbases

Definition 2 *We say that a dyadic subbase is proper if* $\mathrm{cl}\, S(d) = \bar{S}(d)$ *for every* $d \in K(\mathbb{T}^\omega)$.



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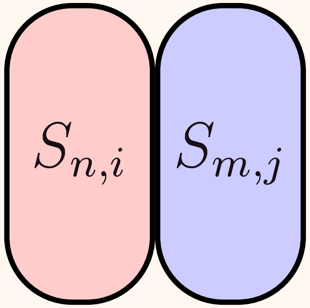

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(1) *If* $x \neq y \in X$, *then* x *and* y *are separated by* $S_{n,i}$ *and* $S_{n,1-i}$ *for some* n *and* i.

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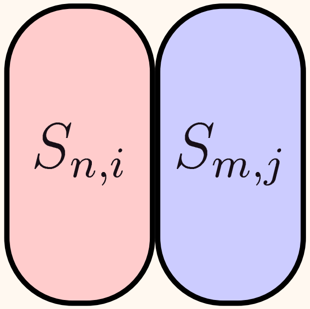

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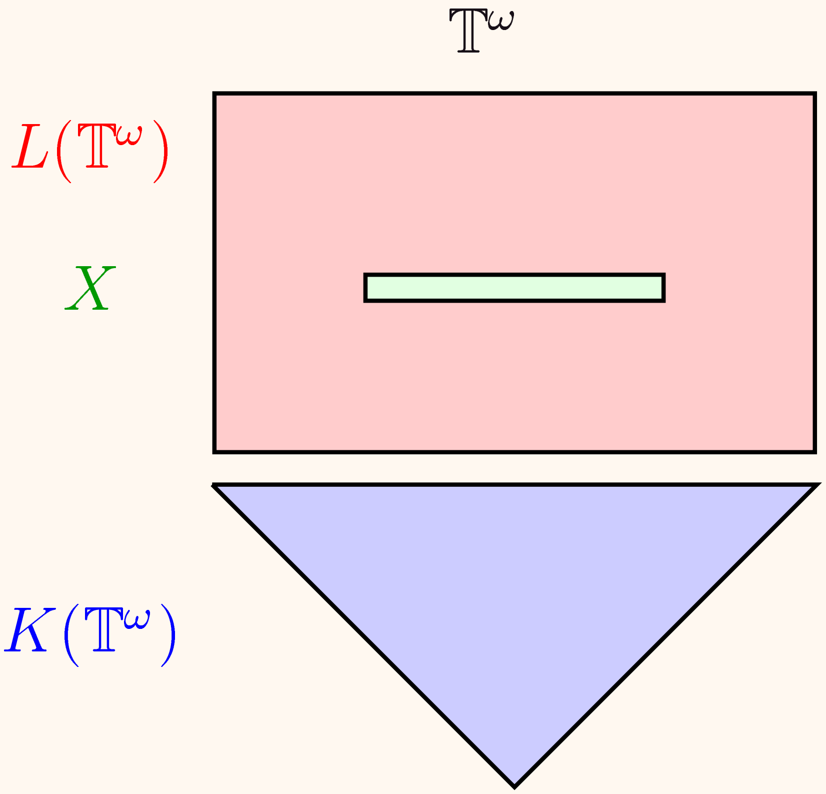

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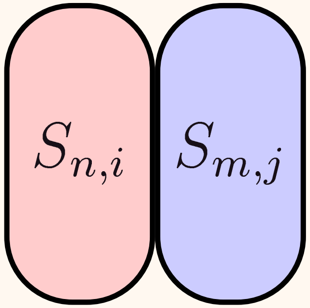

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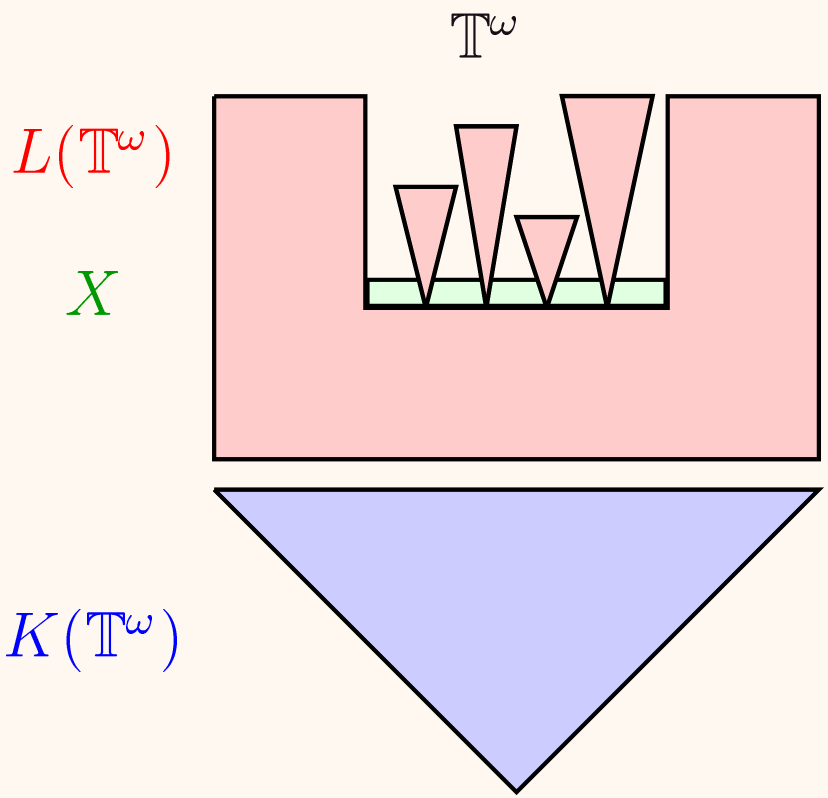

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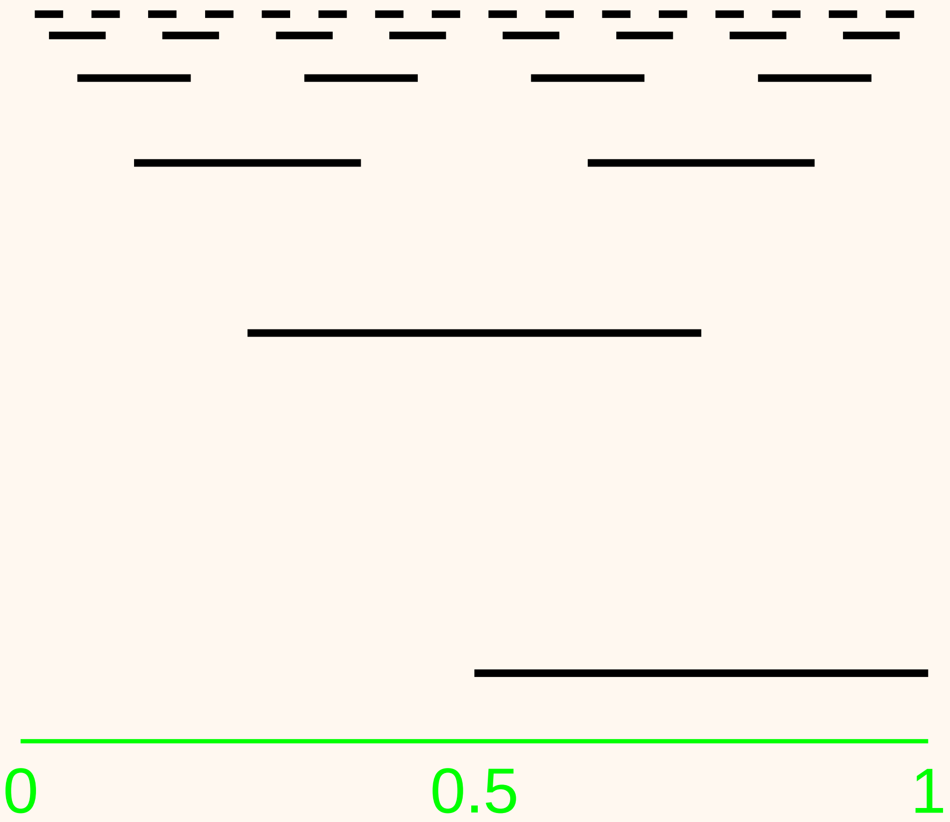

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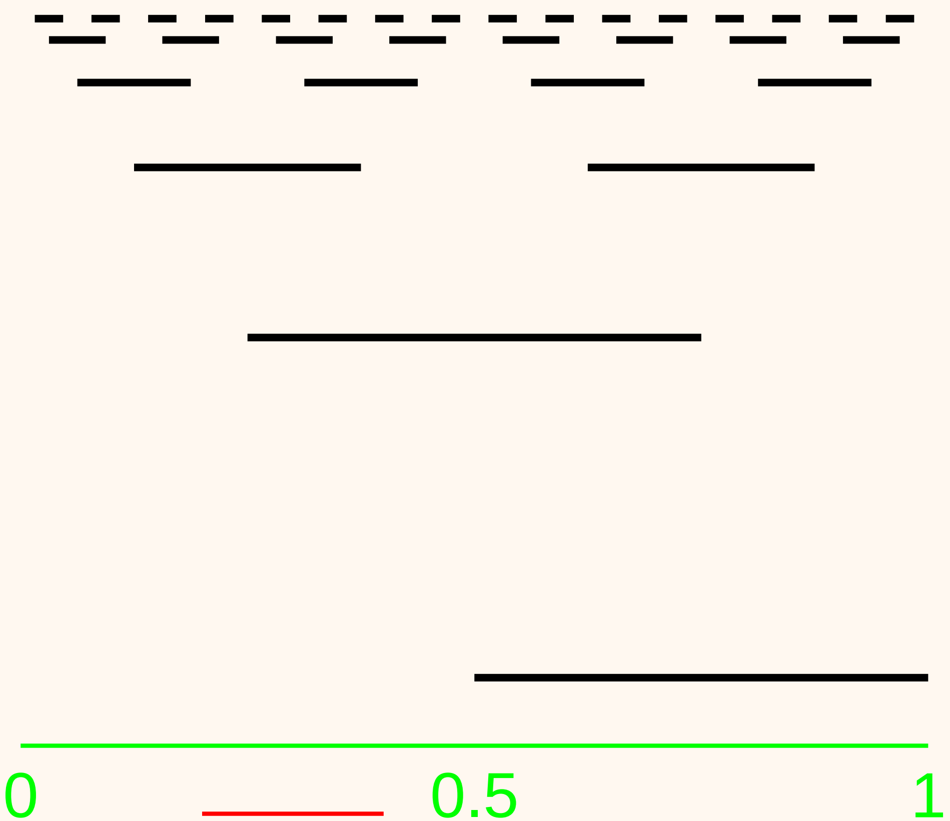

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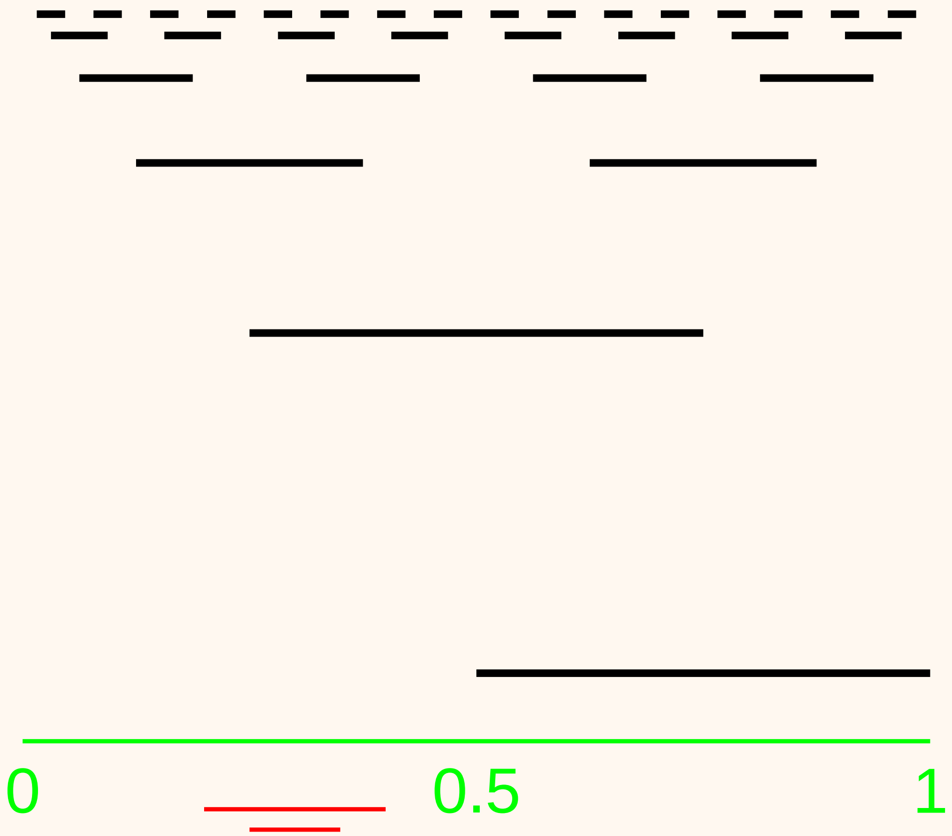

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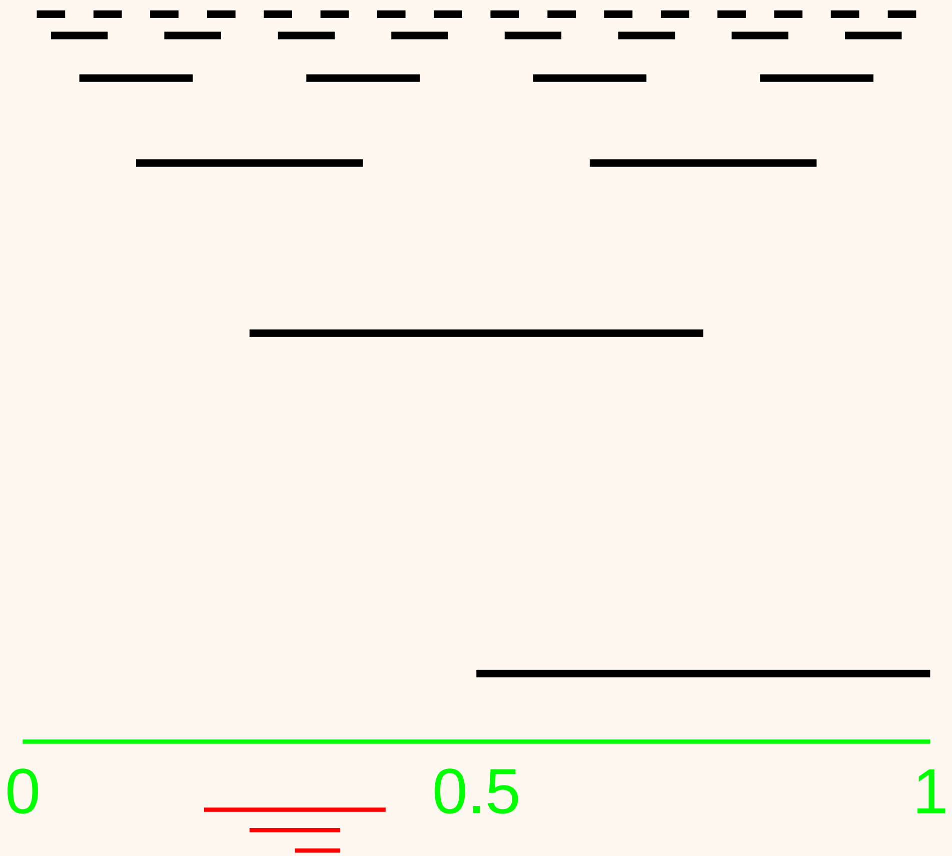

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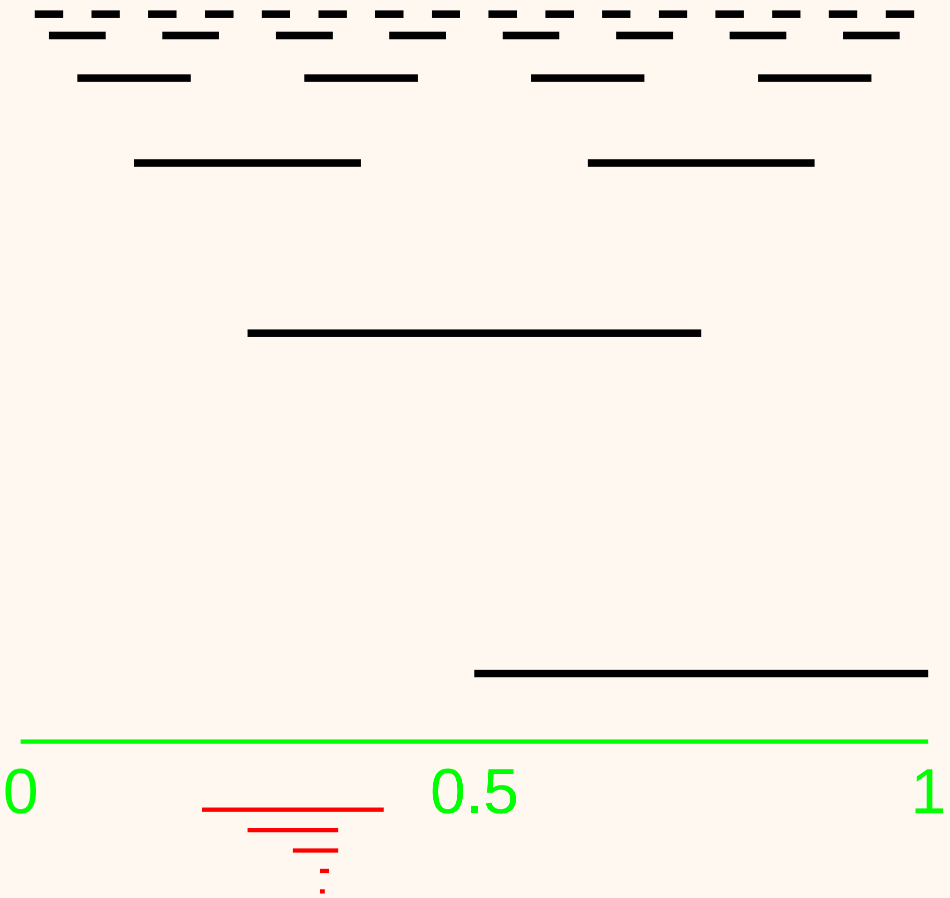

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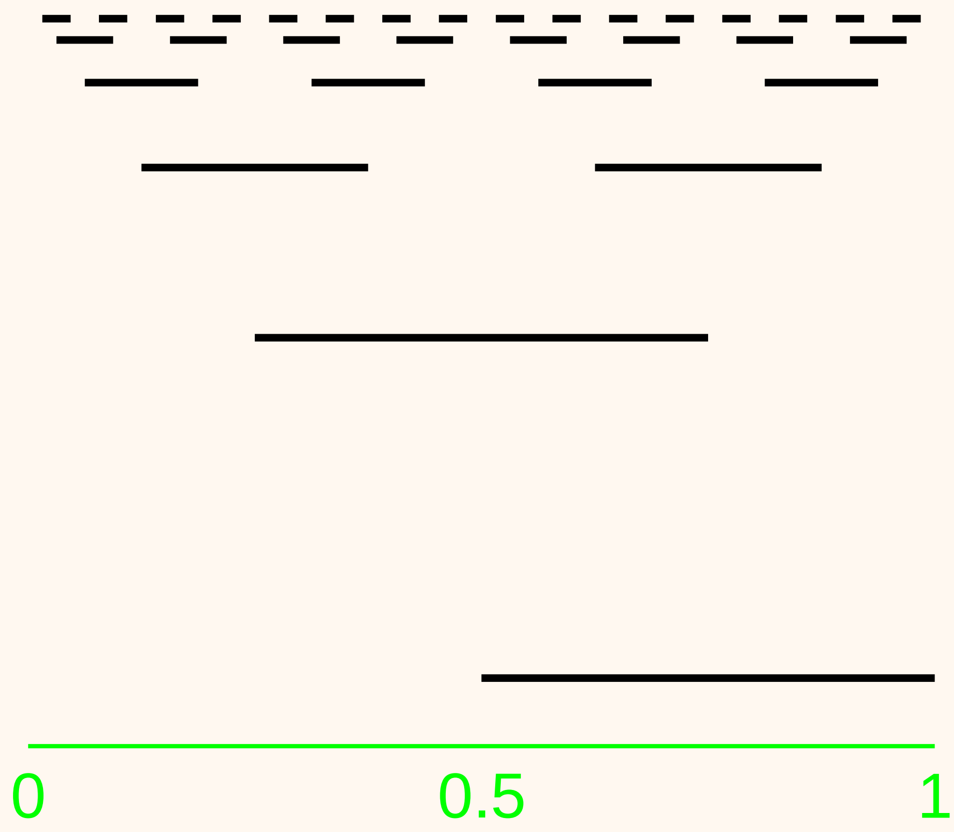

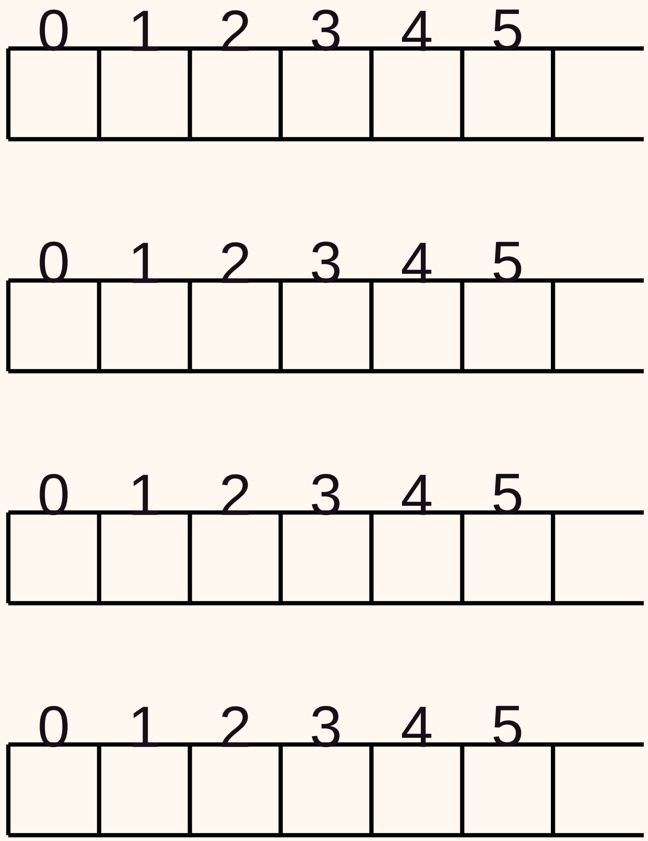

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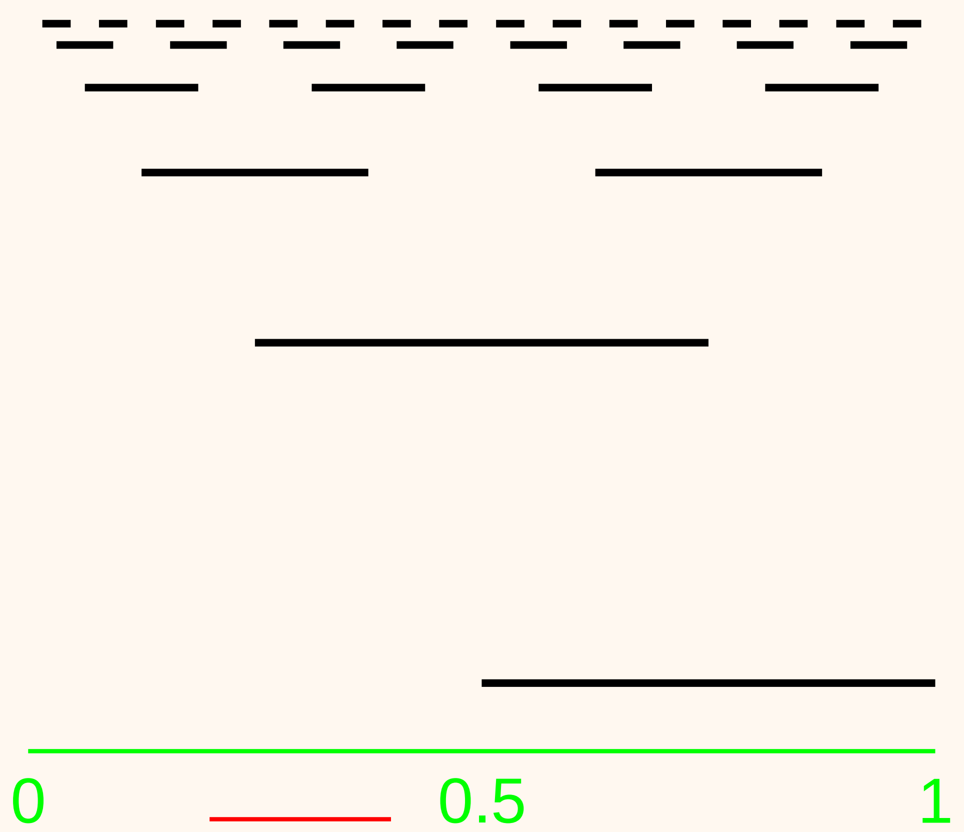

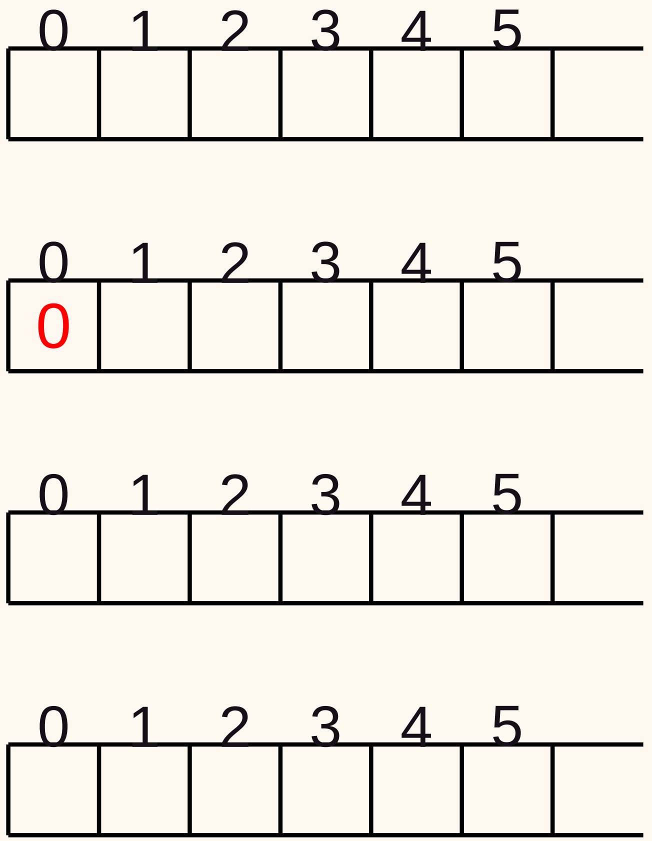

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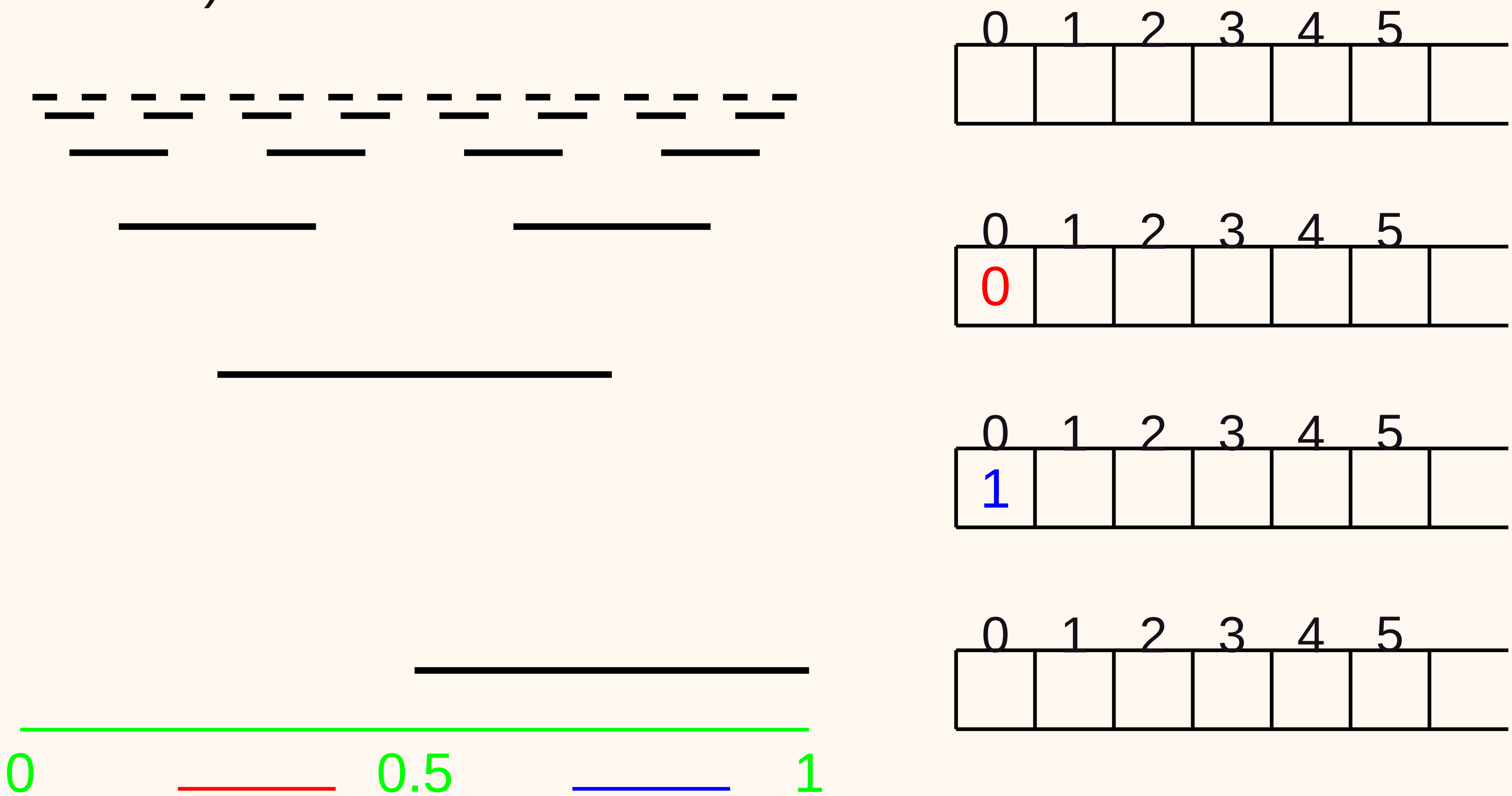

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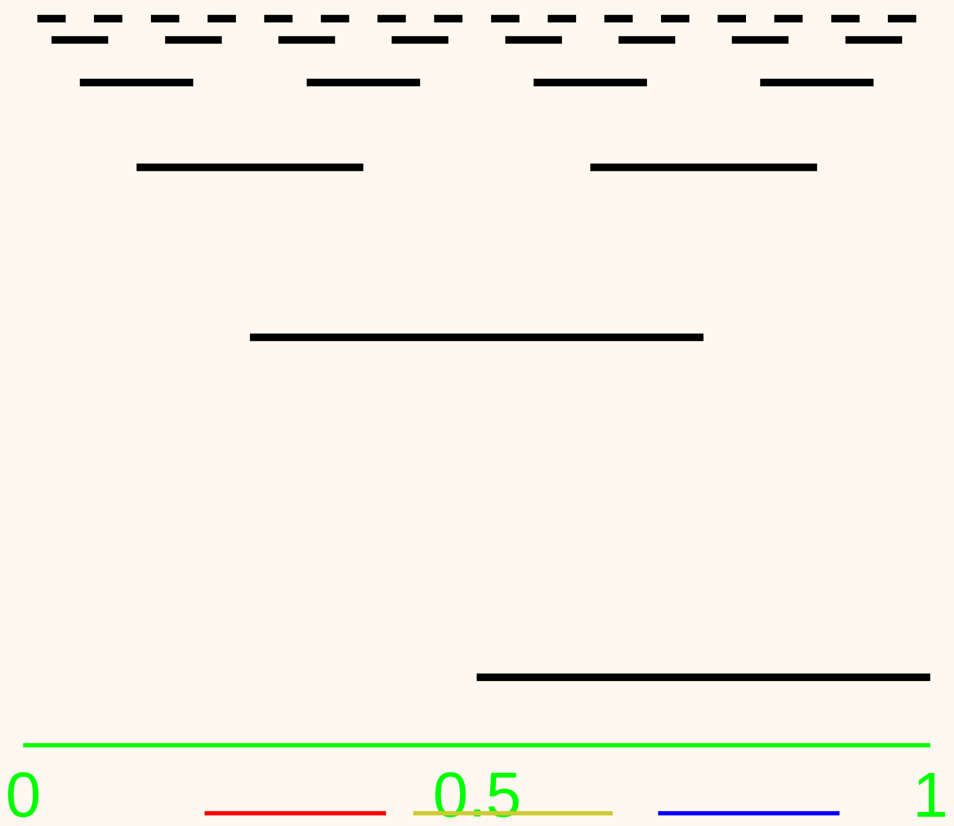

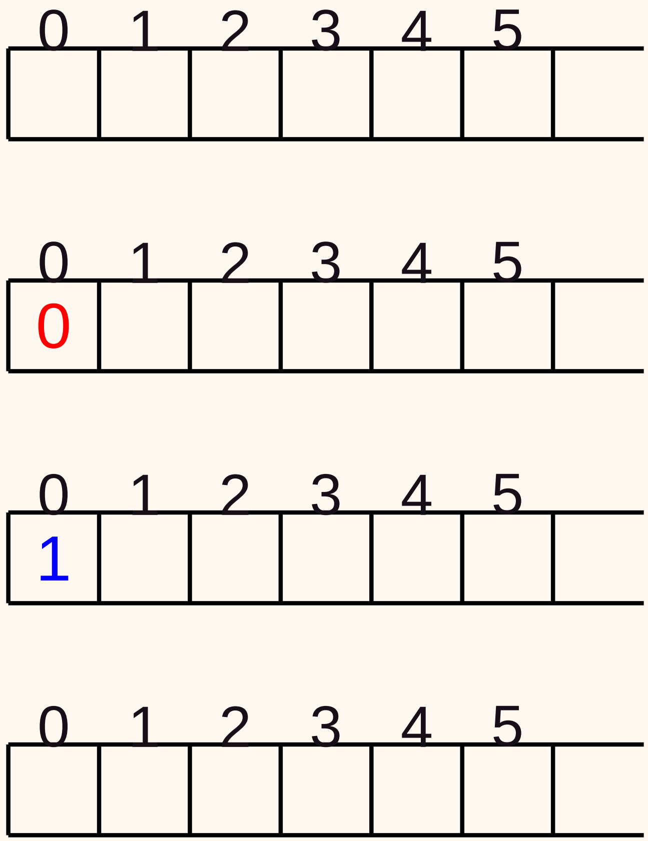

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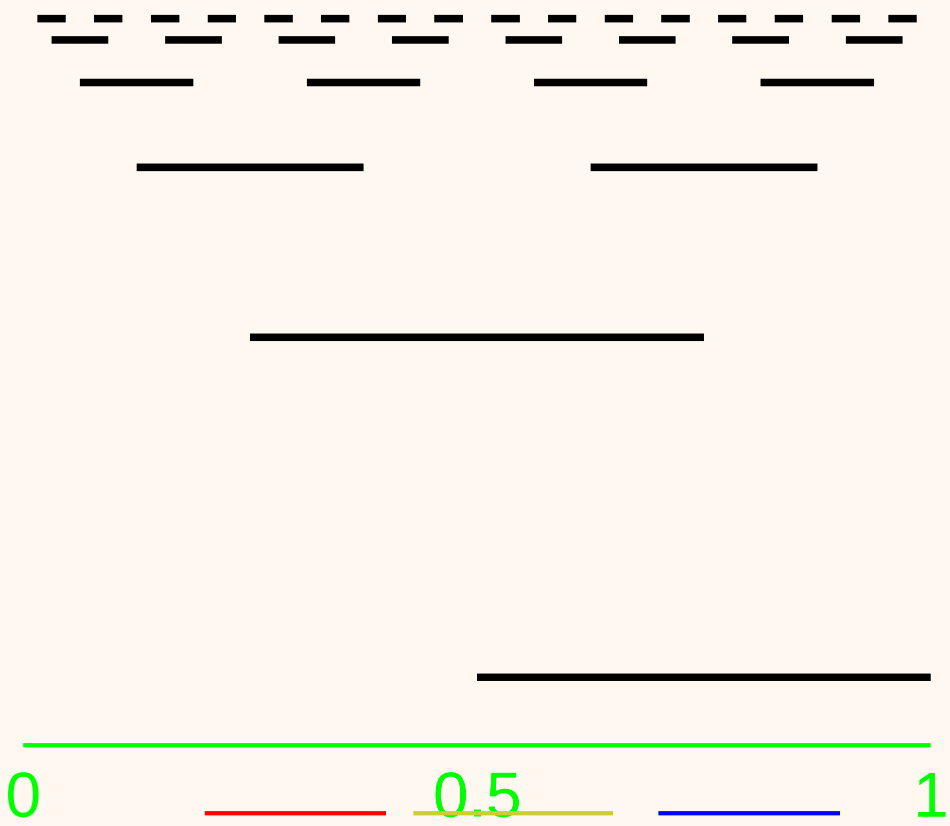

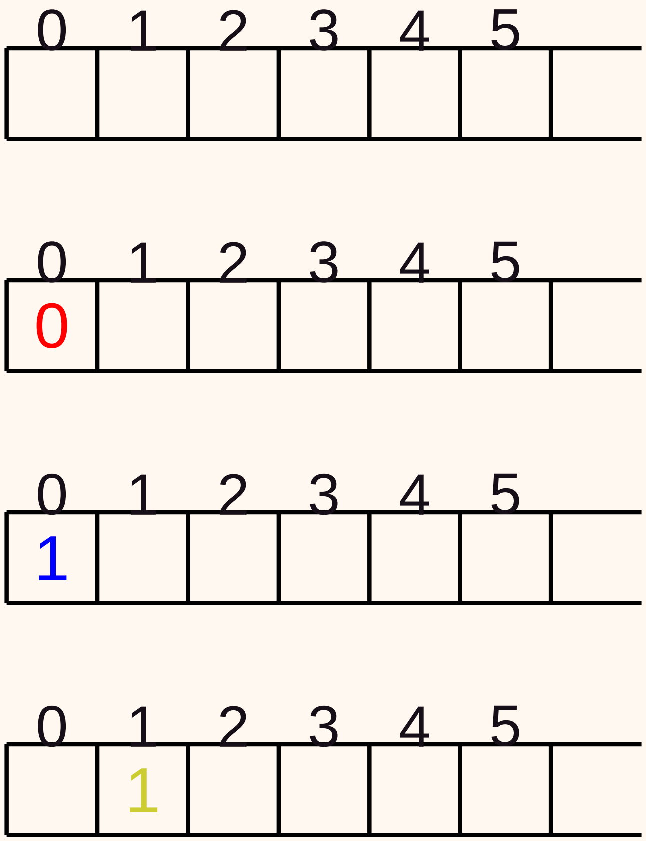

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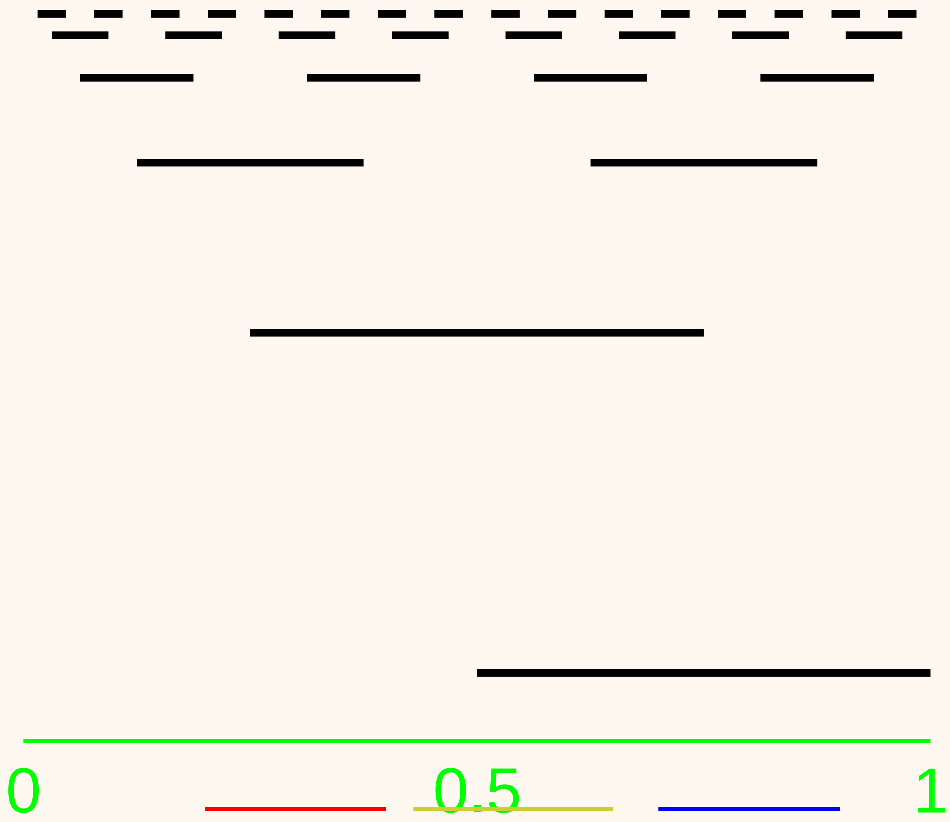

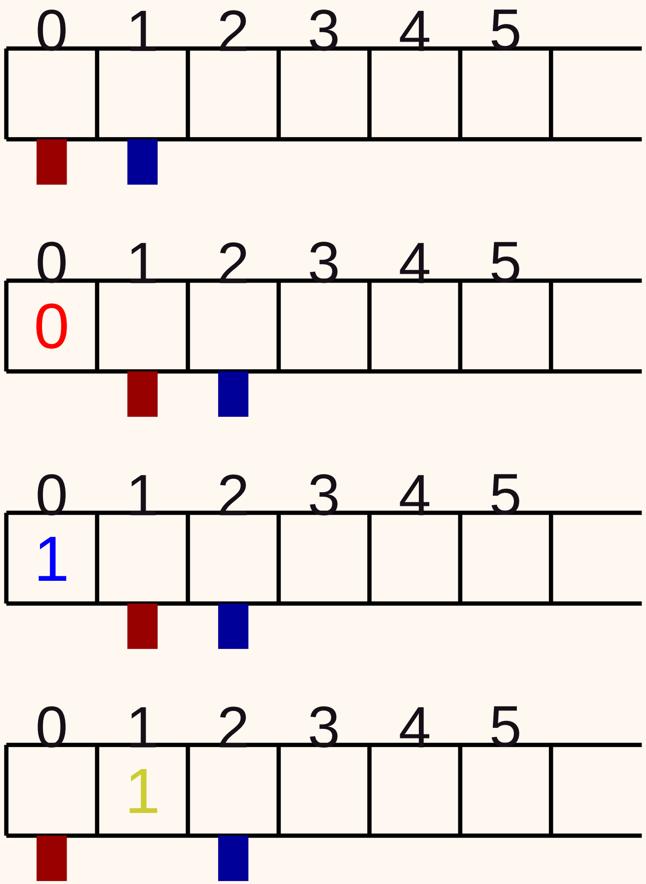

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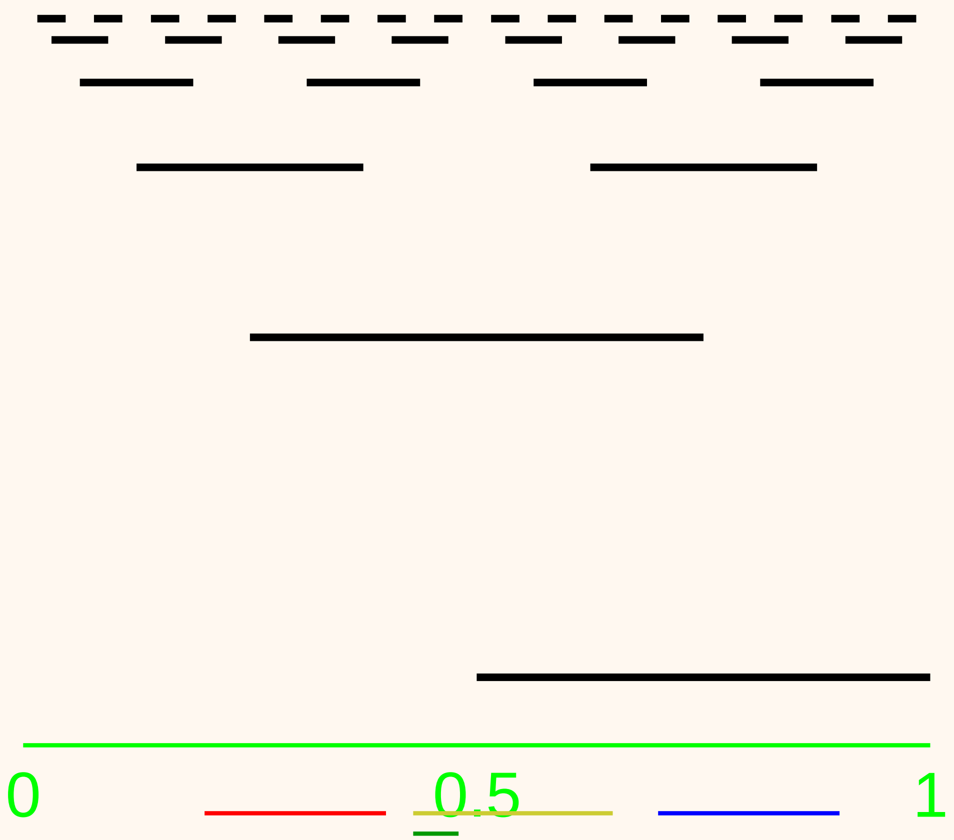

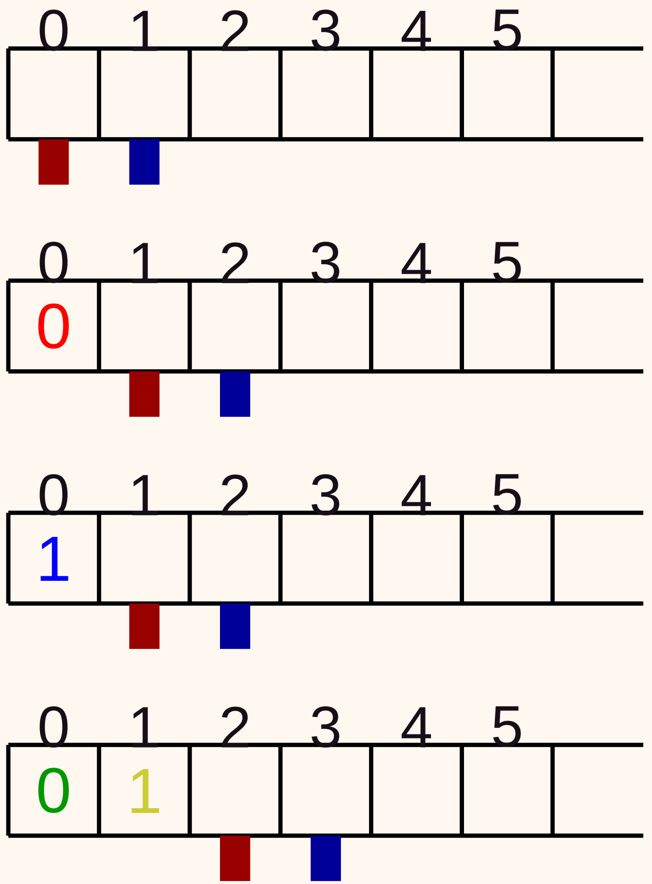

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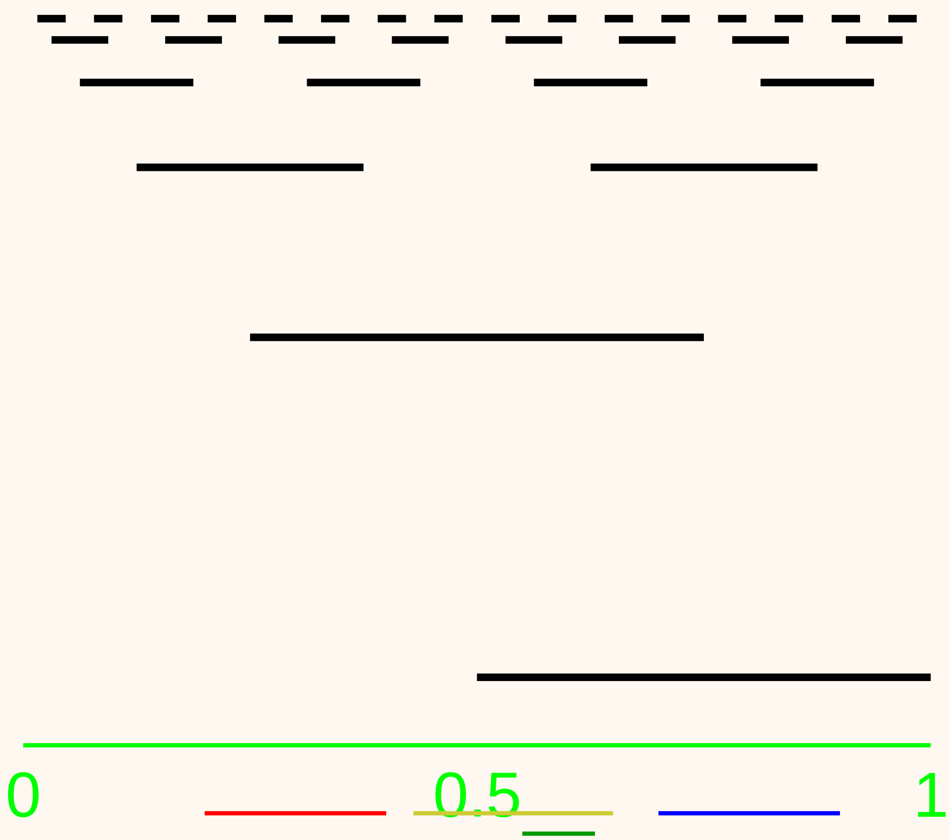

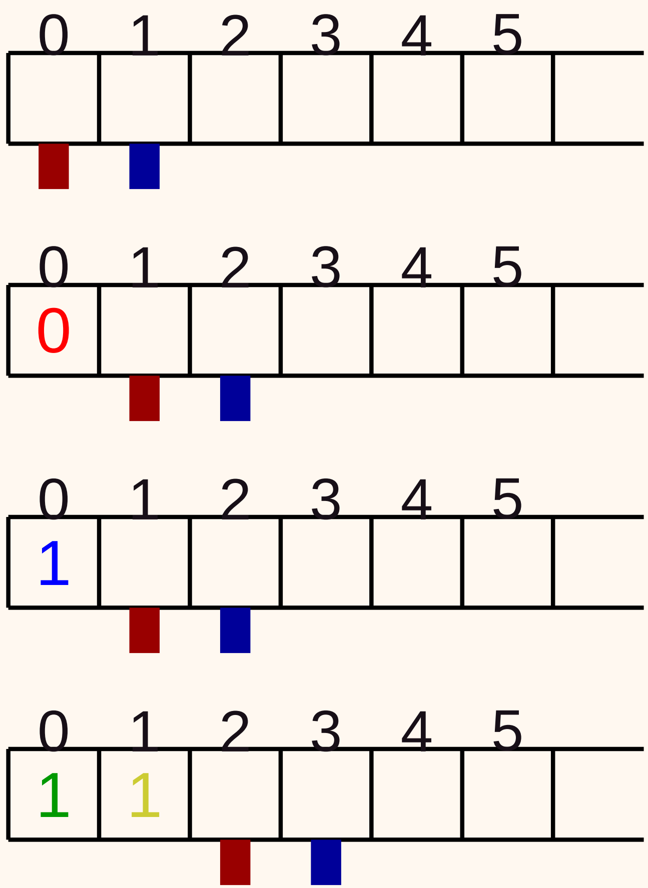

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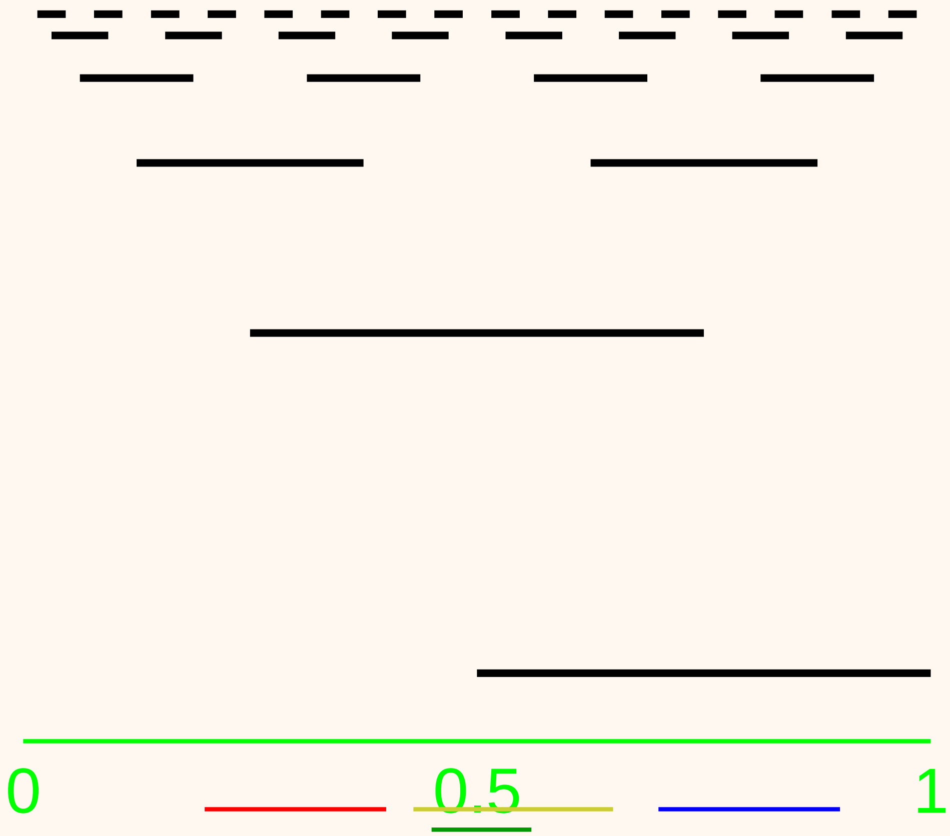

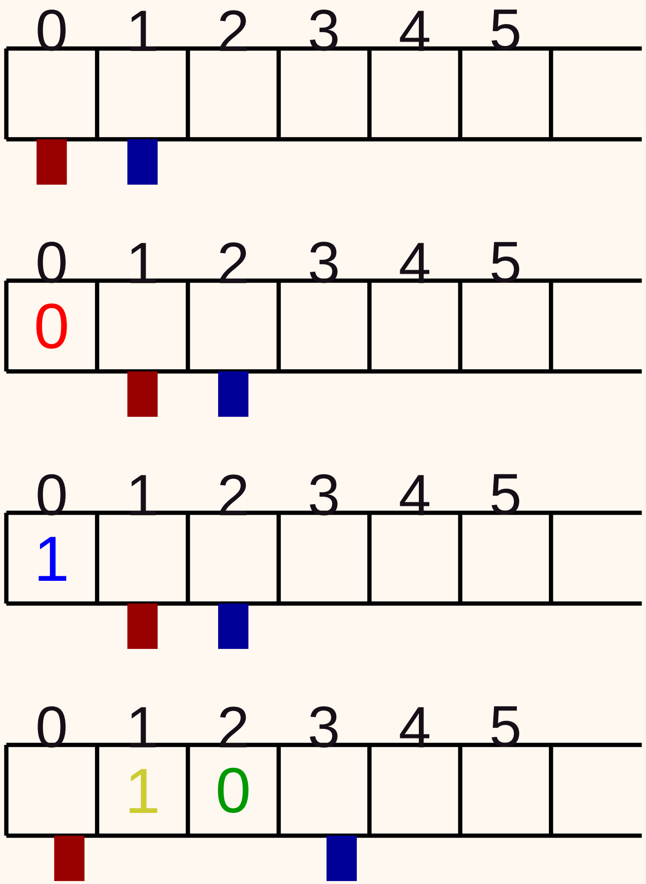

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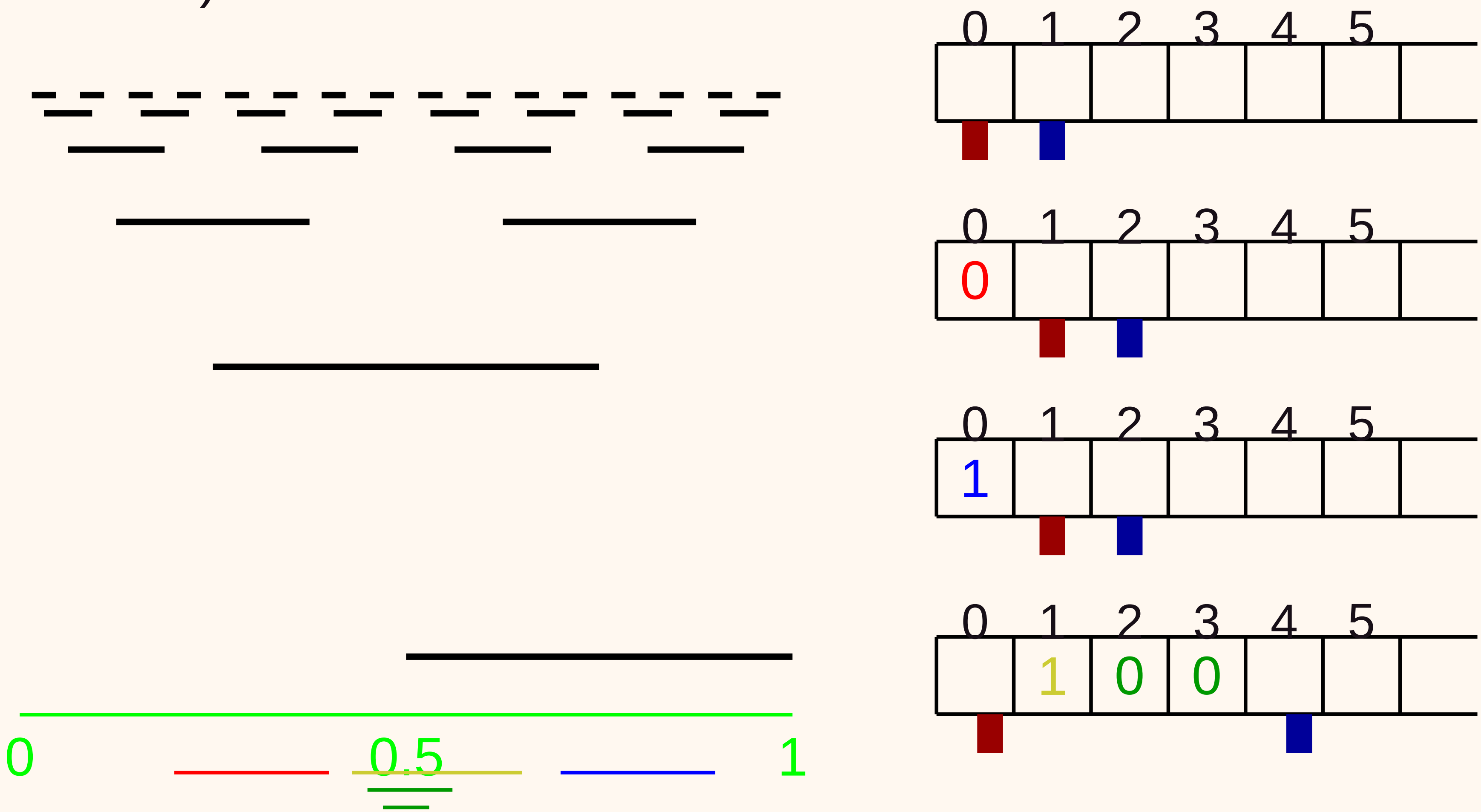

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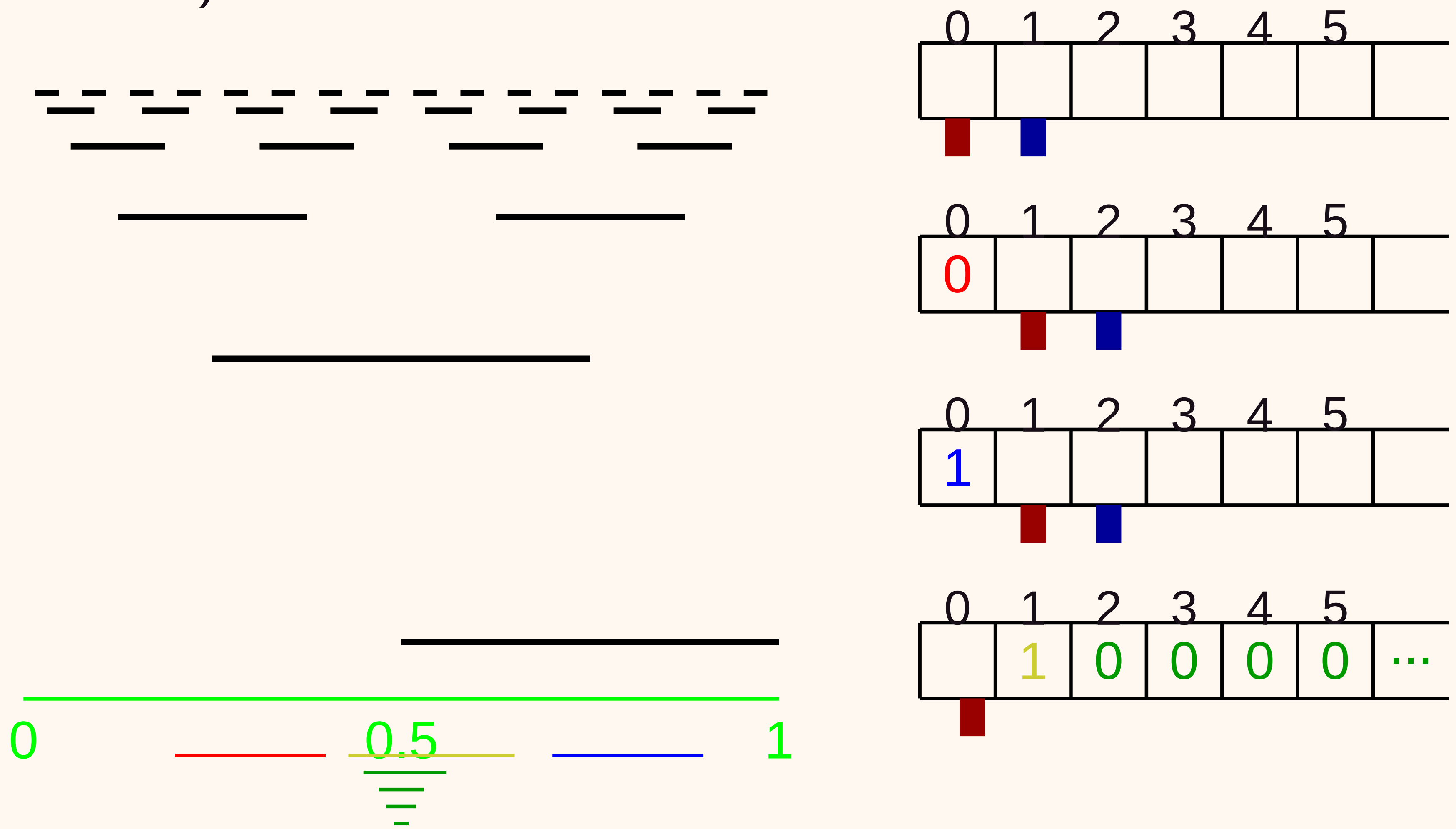

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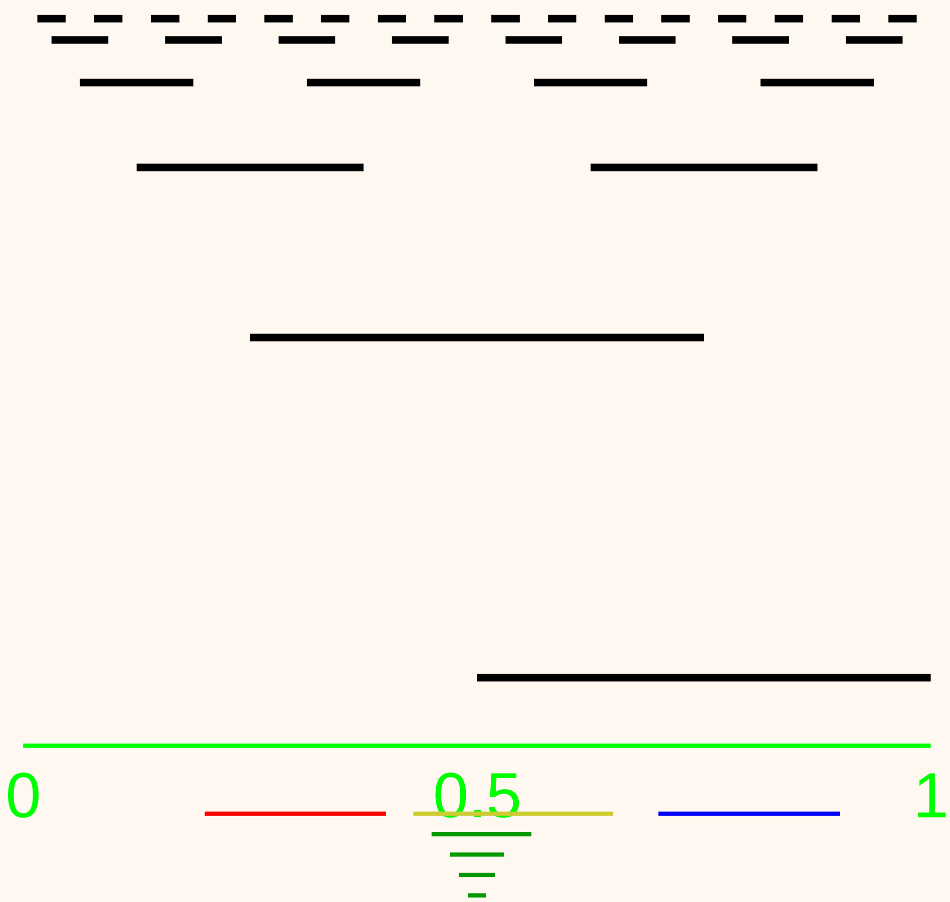

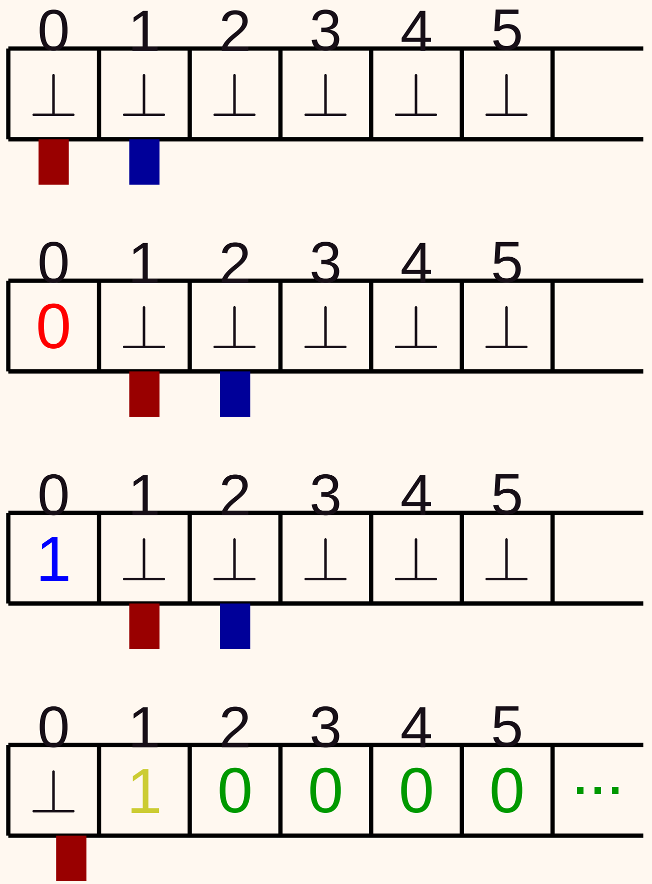

How can we input Gray-code?

0	1	2	3	4	5	
1	0	$\perp$	1	0	0	

0	1	2	3	4	5	

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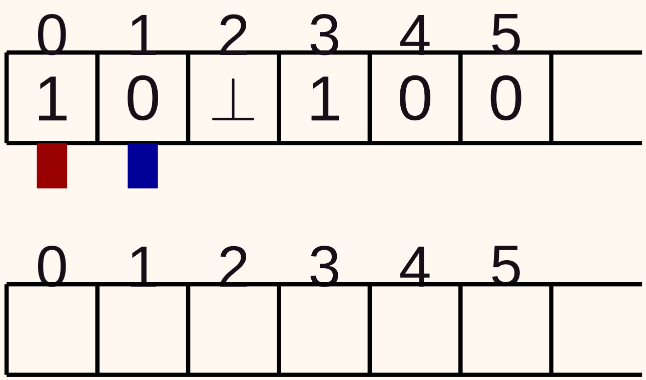

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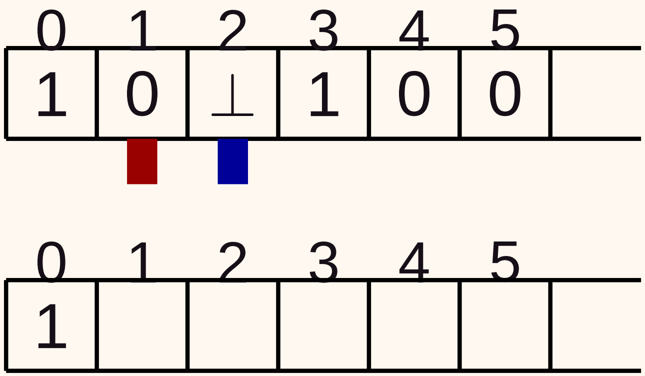

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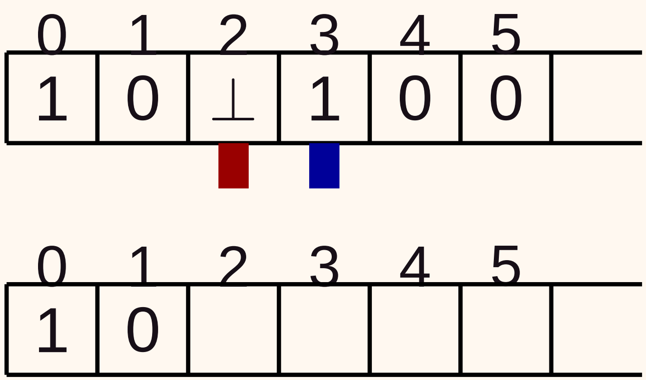

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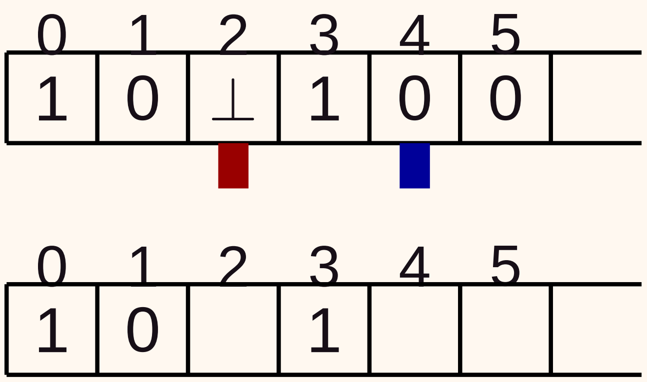

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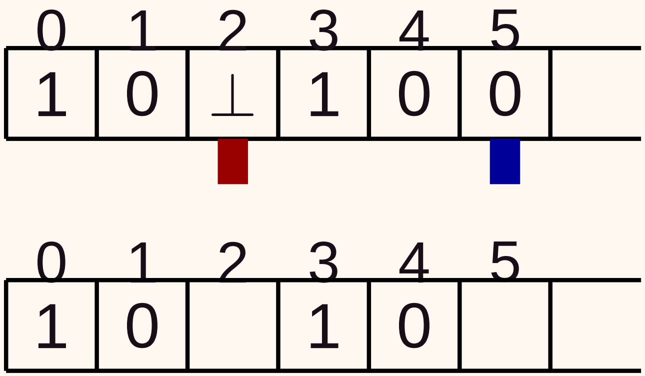

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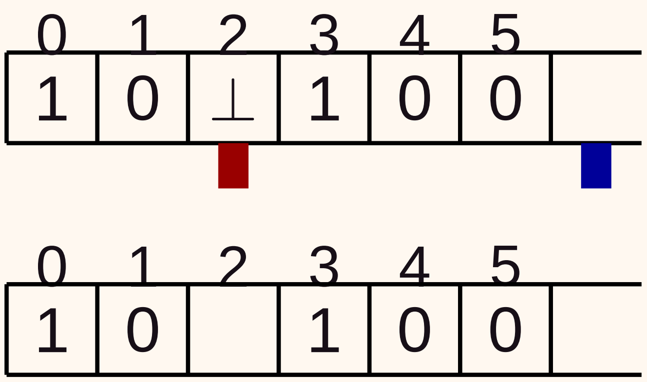

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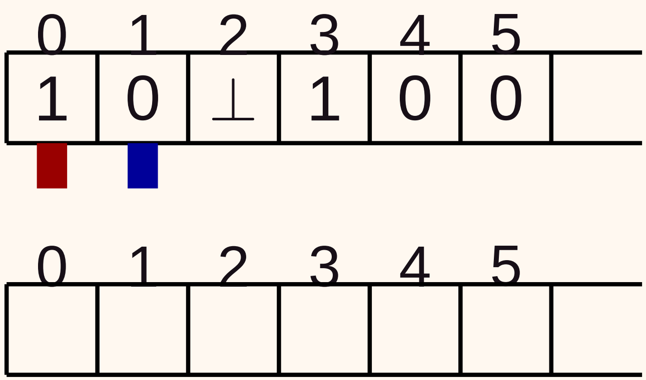

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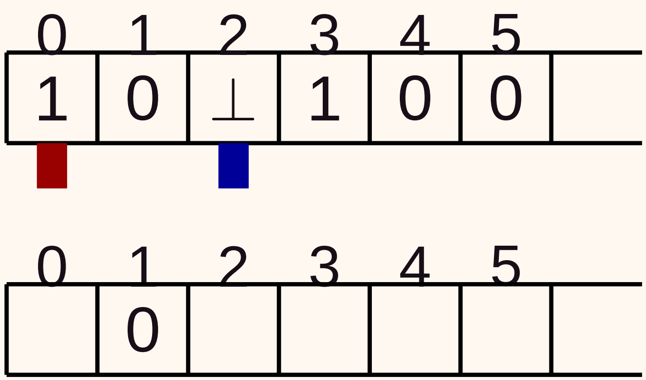

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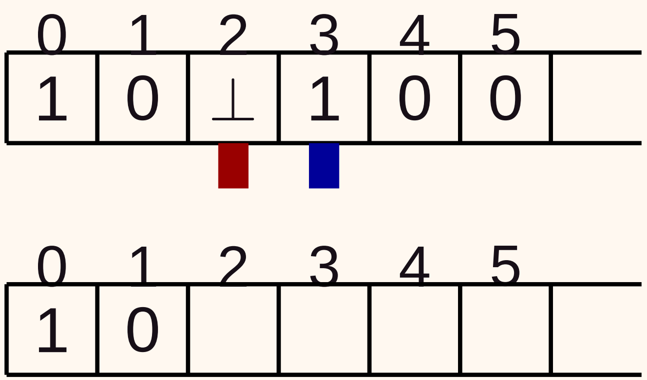
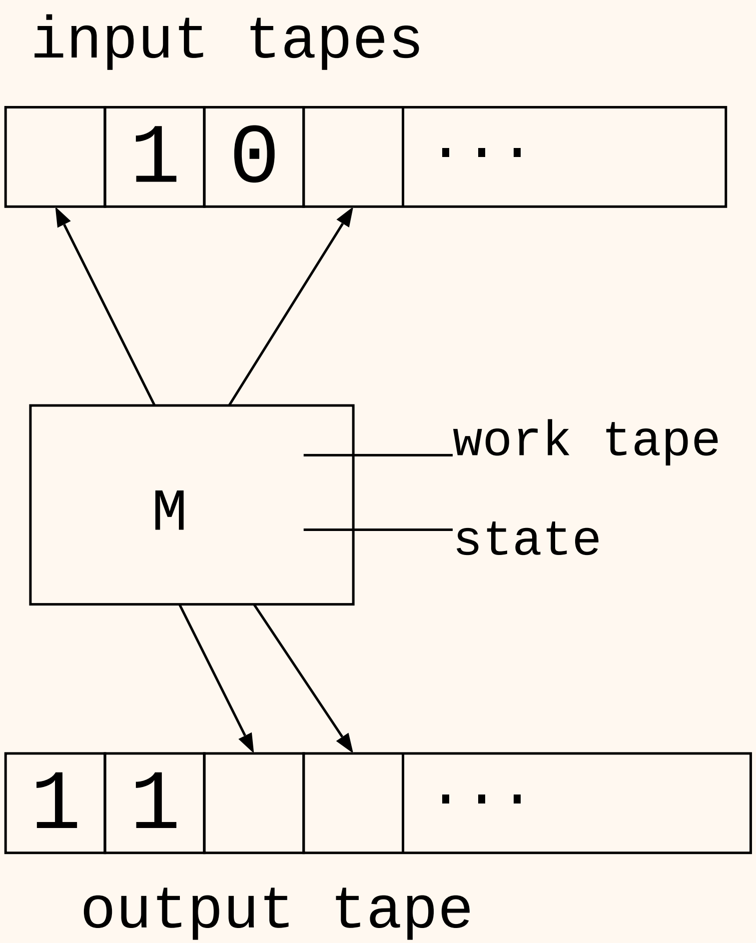


We may input from blue head first.

Two possible inputs if we have digits on both heads.

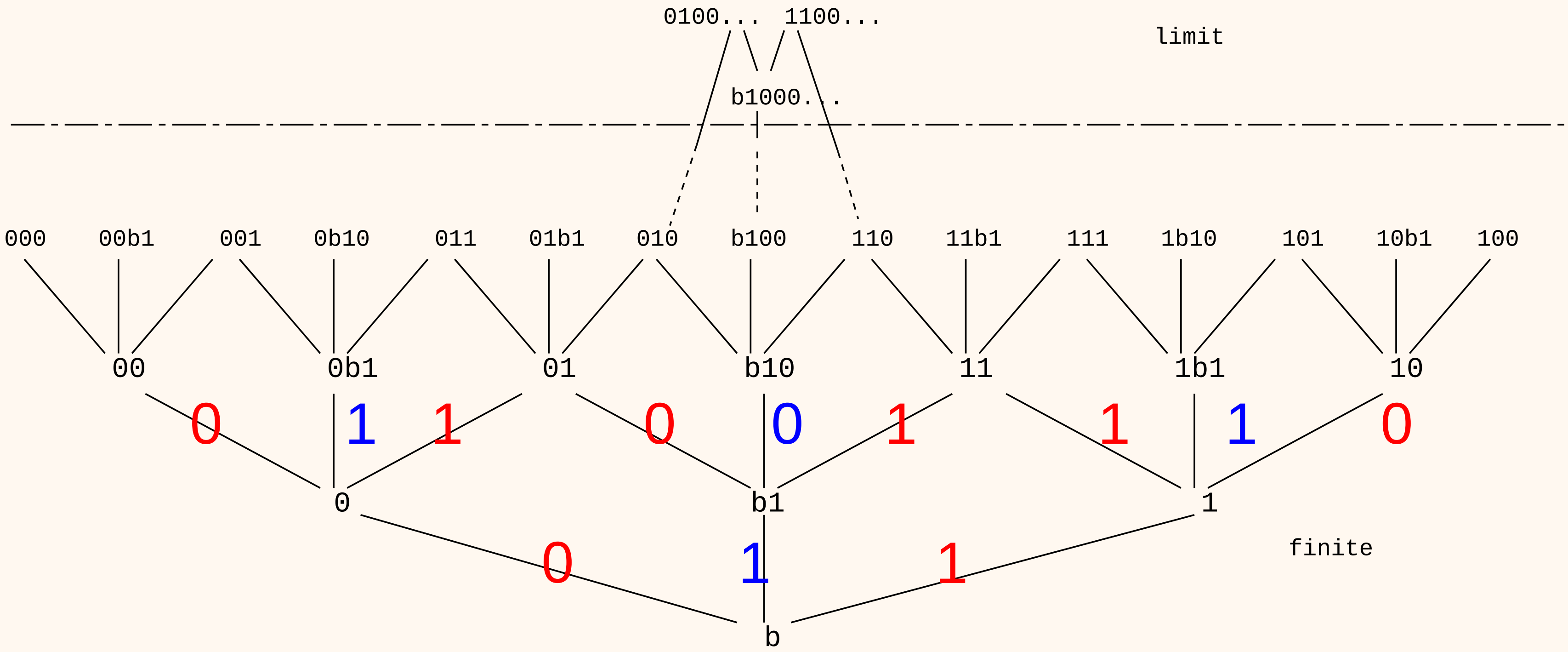
$\Rightarrow$ Indeterministic (non-deterministic) behavior.

IM2-machine



- Generalization of Type-2 machine with 2-heads input/output access.
- Indeterministic (i.e. nondeterministic) behavior depending on the head used to input.
  $\rightarrow$ defines a multi-valued function.
  note: Multi-valuedness is essential for real number computation

The Poset of Finite/Infinite states



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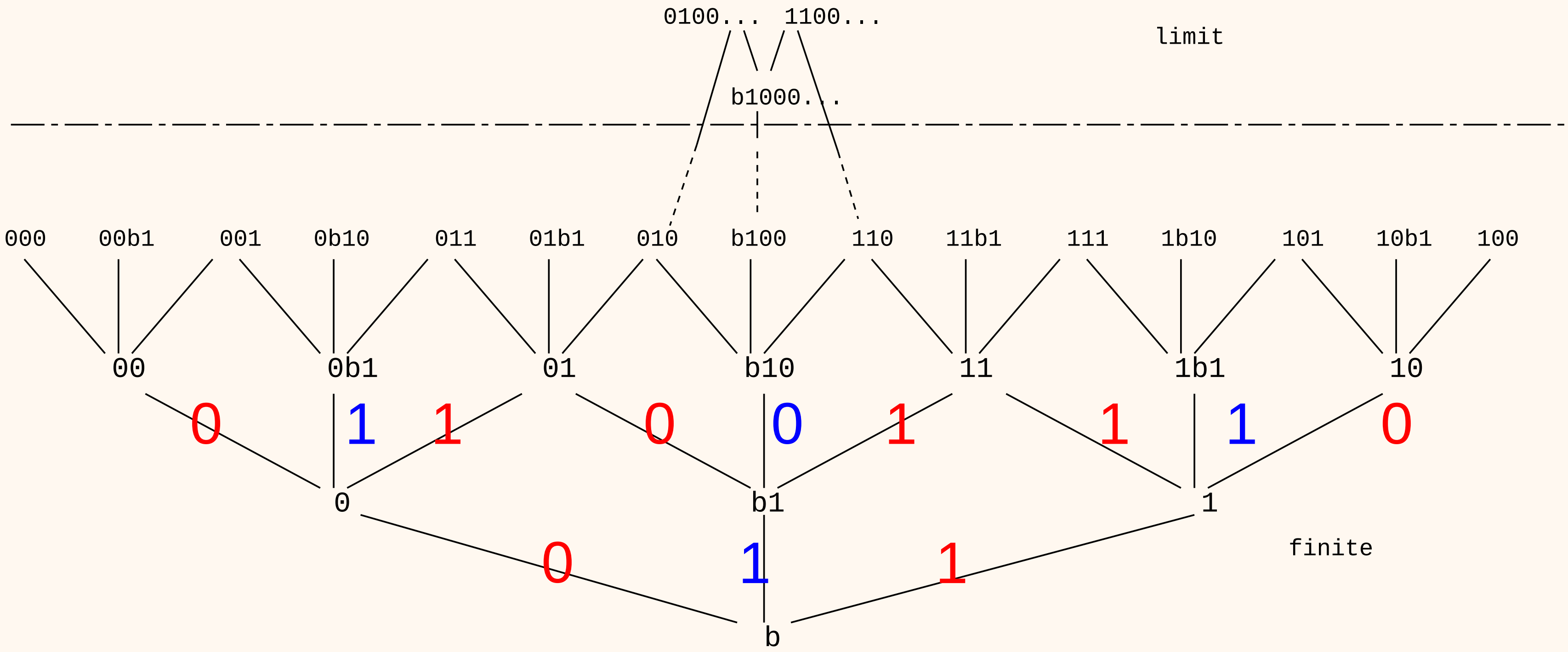

- Infinite-State: obtained as ideal completion of the finite states.

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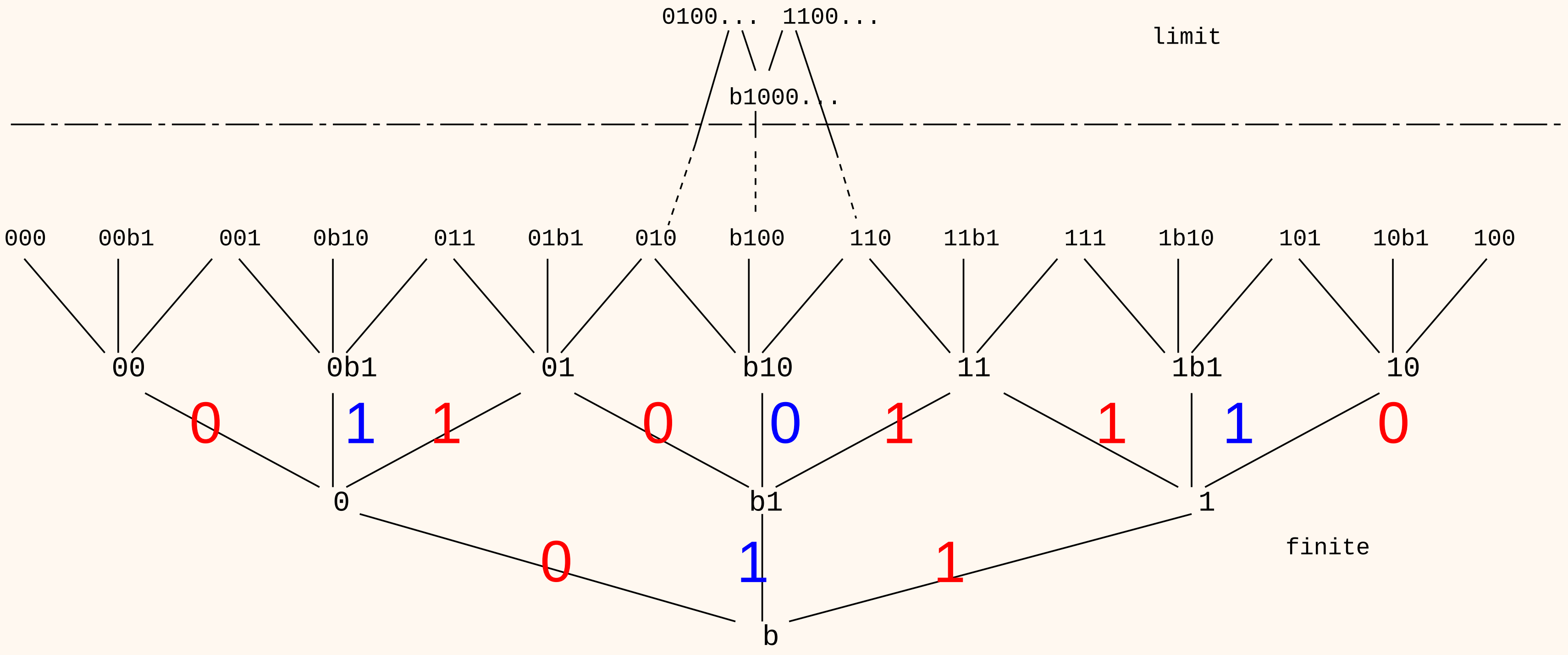

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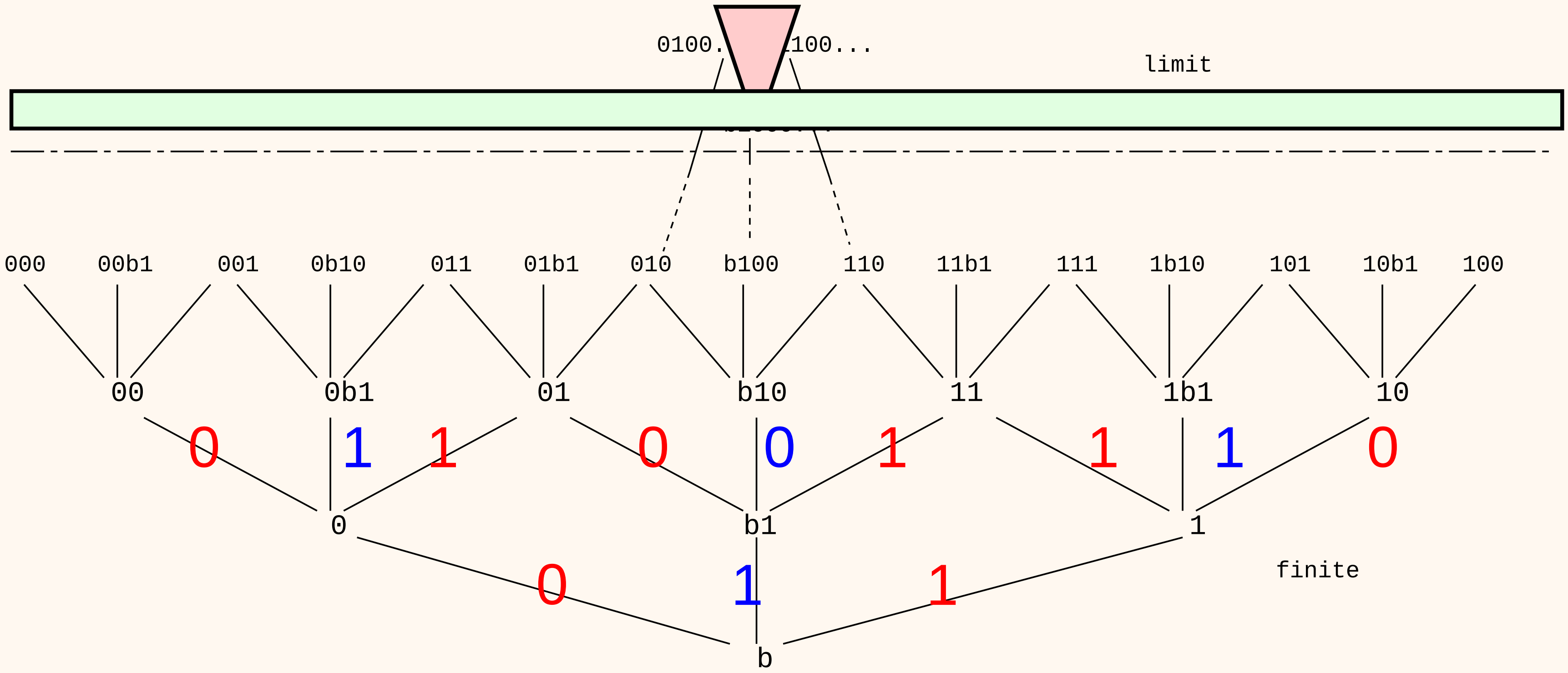

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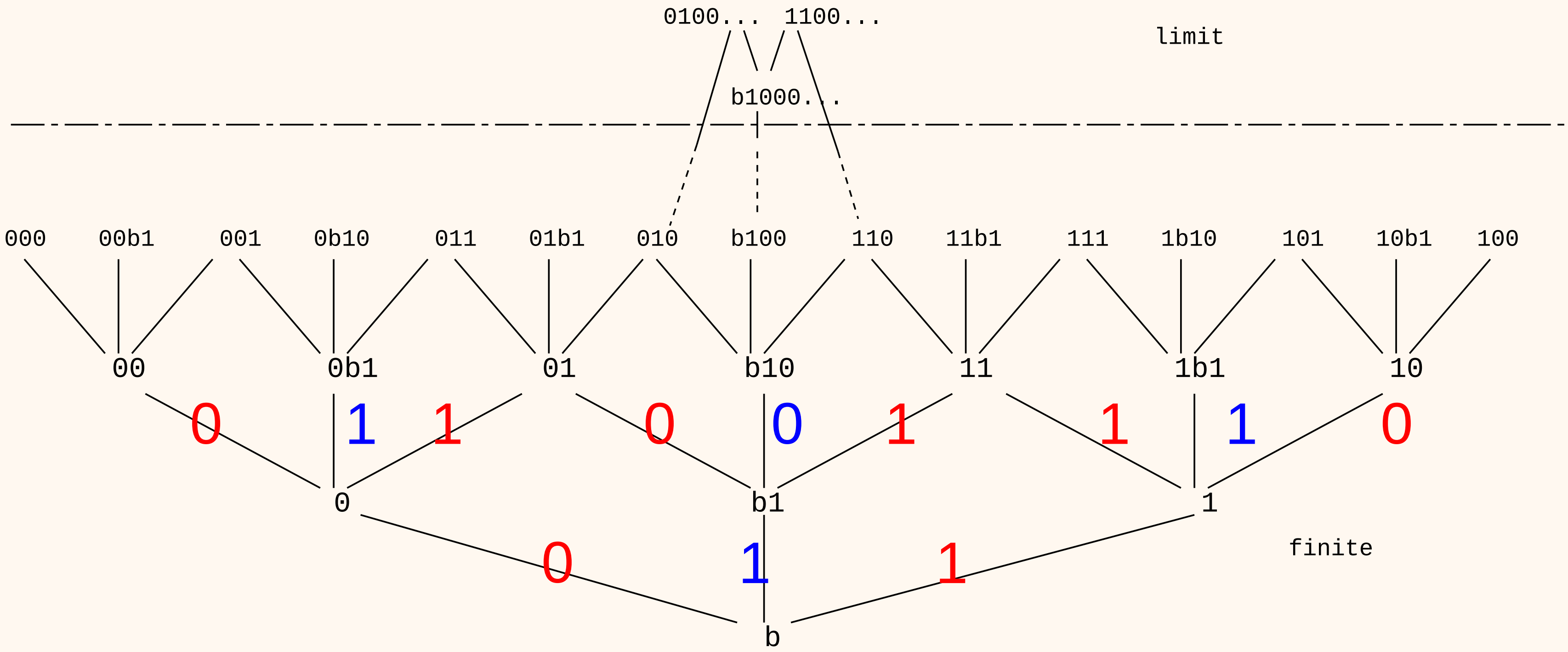


- Infinite-State: obtained as ideal completion of the finite states.
- Subdomain of $\mathbb{T}^{\omega}$.
- The set of minimal elements of the limit elements homeomorphic to $\mathbb{I}$.
- The same domain as that of Signed Digit Representation.
- Admissible Representation of $\mathbb{I}$.

Computation over Topological Spaces

Embedding in $\mathbb{T}^\omega$ for $\mathbb{T} = \{0, 1, \perp\}$.
$\cdots$ subspace of a code space with bottoms.

This work started with

- (1) Gray-embedding of the unit interval [0,1]
- (3) IM2-machines which are working on (subsets of) $\mathbb{T}^\omega$.

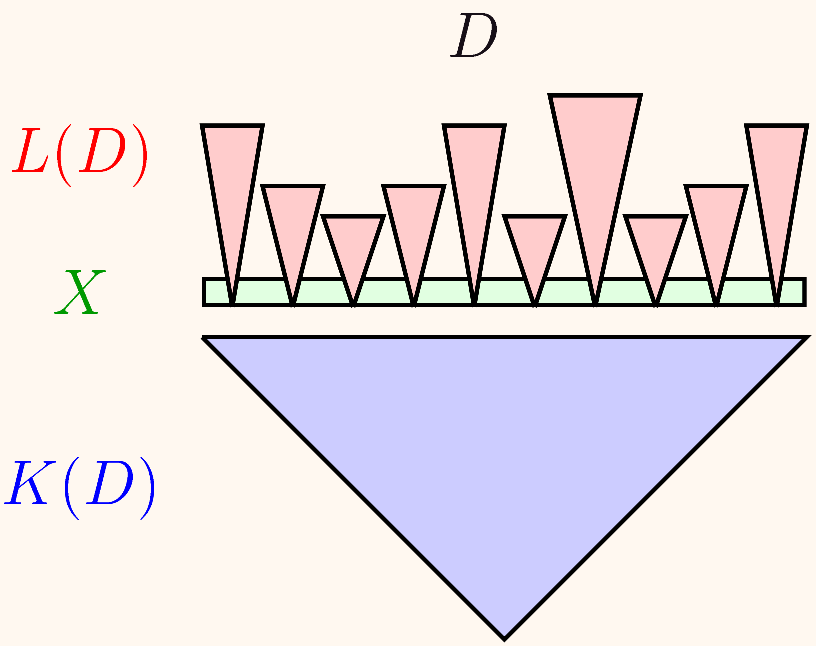
As generalization, we studied

- (2) Dyadic subbase,
- (4) Domain representation as minimal limit sets,
- (6) Uniform domain and uniform space.

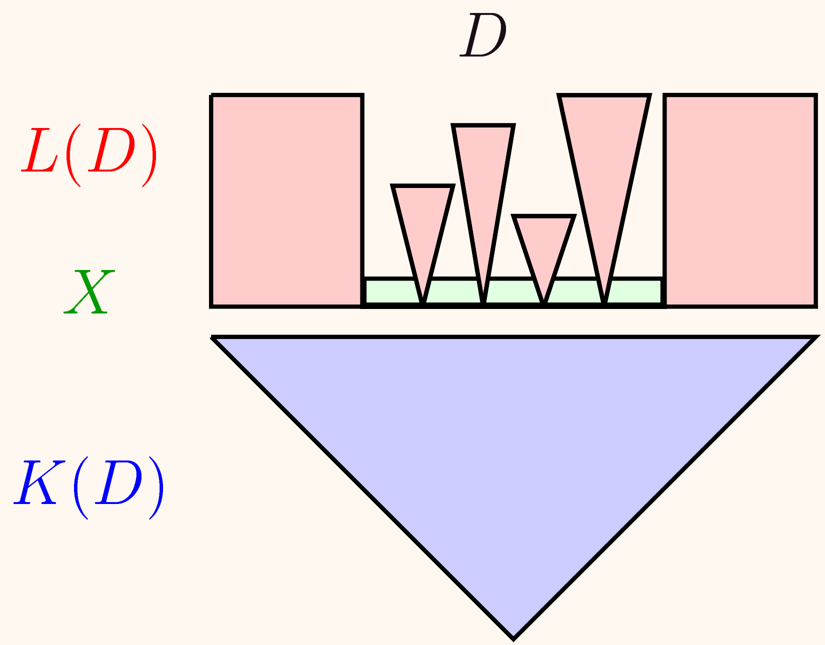
In this talk, we connect them. that is,

- (5) Construct a domain representation from a dyadic subbase.
- (7) Derive uniformity structure from it.

Minimal-Limit Sets of a Domain



Compact Case



Non-Compact Case.

- D is an algebraic subdomain of T^ω.
- $L(D)$ has enough minimal elements.
  (for all $q \in P$, exists a minimal p s.t. $p \leq q$.)
- X is densely embed in $\min(L(D))$ (and $L(D)$ and D).
- We can derive an admissible representation of X.

Our Goal

Construct such a domain representation from a dyadic subbase of X.

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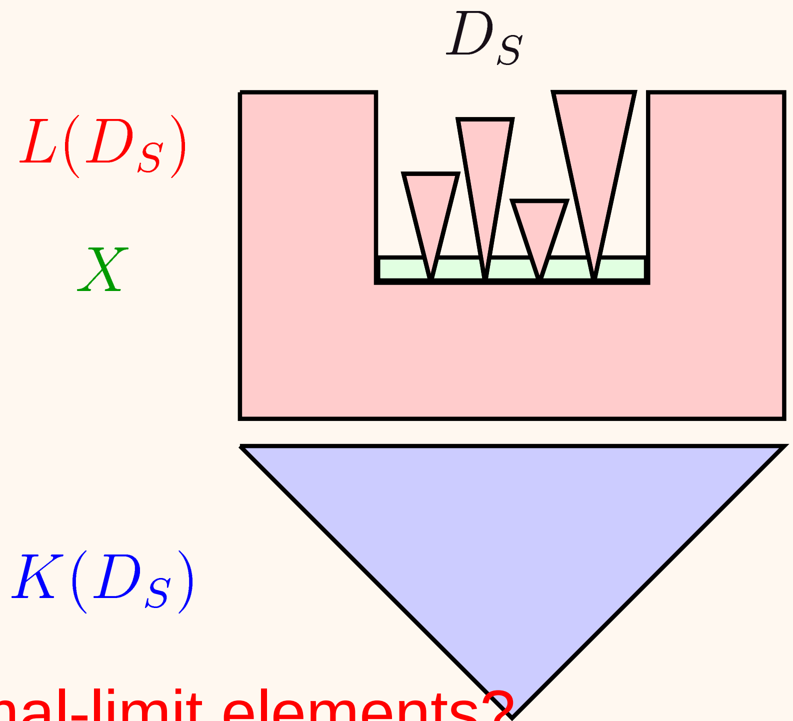
Domain D_S

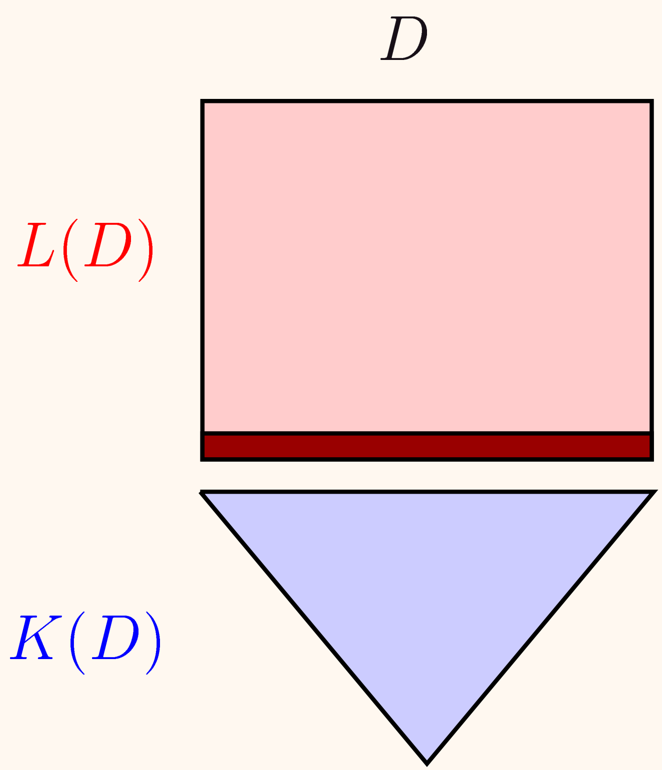
- Let X be a Hausdorff space and $S = \{S_{n,i} : n < \omega, i < 2\}$ be a proper dyadic subbase of X.
- For $p \in \mathbb{T}^\omega$, let $p_{<m} \in K(\mathbb{T}^\omega)$ be $p_{<m}(n) = p(n)(n < m)$ and $p_{<m}(n) = \bot(n \geq m)$.
- $K_S = \{\varphi_S(x)_{<m} : x \in X, m \in \mathbb{N}\} \subset K(T^\omega)$.
- $D_S =$ the ideal completion of K_S.
- D_S is a subdomain of $\mathbb{T}^\omega$.
- $K_S = K(D_S)$.
- $\varphi_S(X) \subset L(D_S)$.



When does X become the set of minimal-limit elements?

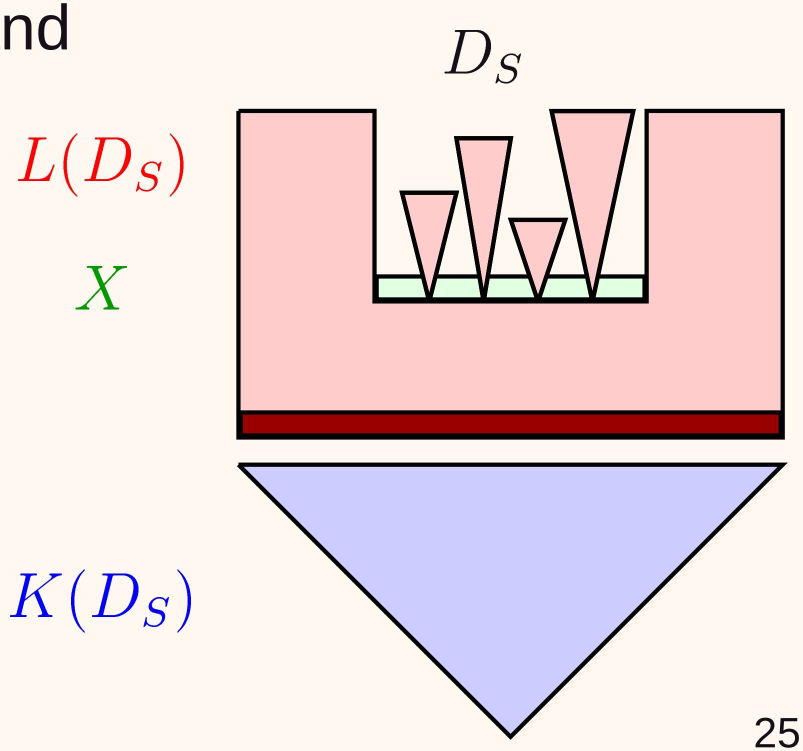
Finite-Branching Domain

Theorem 5 *Suppose that $K(D)$ is finite-branching.*
(1) $L(D)$ is compact.
(2) $L(D)$ has enough minimal elements.
(3) $\min(L(D))$ is compact.



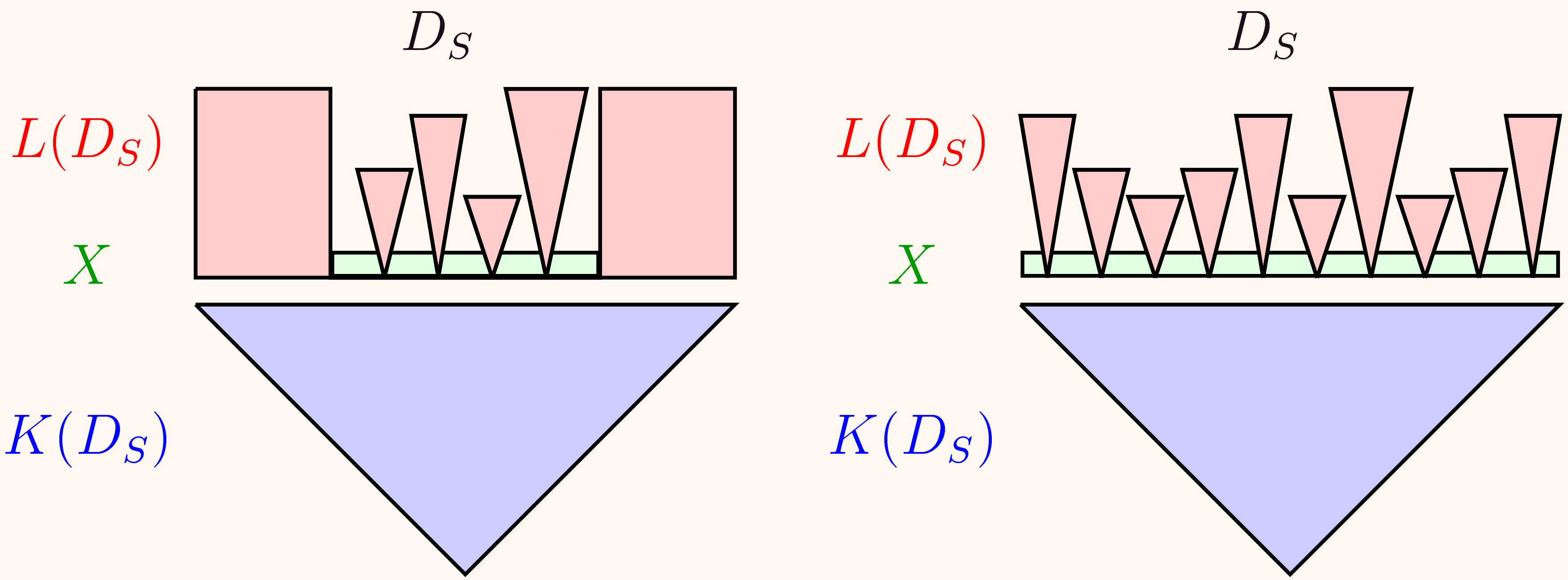
Adhesive Space, $\mathrm{T}_{2\frac{1}{4}}$ space.

- **Def.** A space X is adhesive if X has at least two points and closures of any two open sets have non-empty intersection.
- **Note** There is an adhesive Hausdorff space.
- **Def.** A space X is $\mathrm{T}_{2\frac{1}{4}}$ if it is Hausdorff and no open subspace is adhesive.
- **Proposition** A $\mathrm{T}_{2\frac{1}{2}}$ space is $\mathrm{T}_{2\frac{1}{4}}$ and a $\mathrm{T}_{2\frac{1}{4}}$ space is T_2.
- **Proposition** If X is $\mathrm{T}_{2\frac{1}{4}}$, then$K(D_S)$ is finite-branching.
- **Corollary** If X is $\mathrm{T}_{2\frac{1}{4}}$,
  (1) $L(D_S)$ is compact and it has enough minimal elements.
  (2) $\min(L(D_S))$ is compact.



If X is regular, $\varphi_S(X) \subset \min(L(D_S))$

Theorem 6 *(1) If S is a proper dyadic subbase of a* regular *space X, then $\varphi_S(X) \subset \min(L(D_S))$.*
(2) If S is a proper dyadic subbase of a compact regular *space X, then $\varphi_S(X) = \min(L(D_S))$.*



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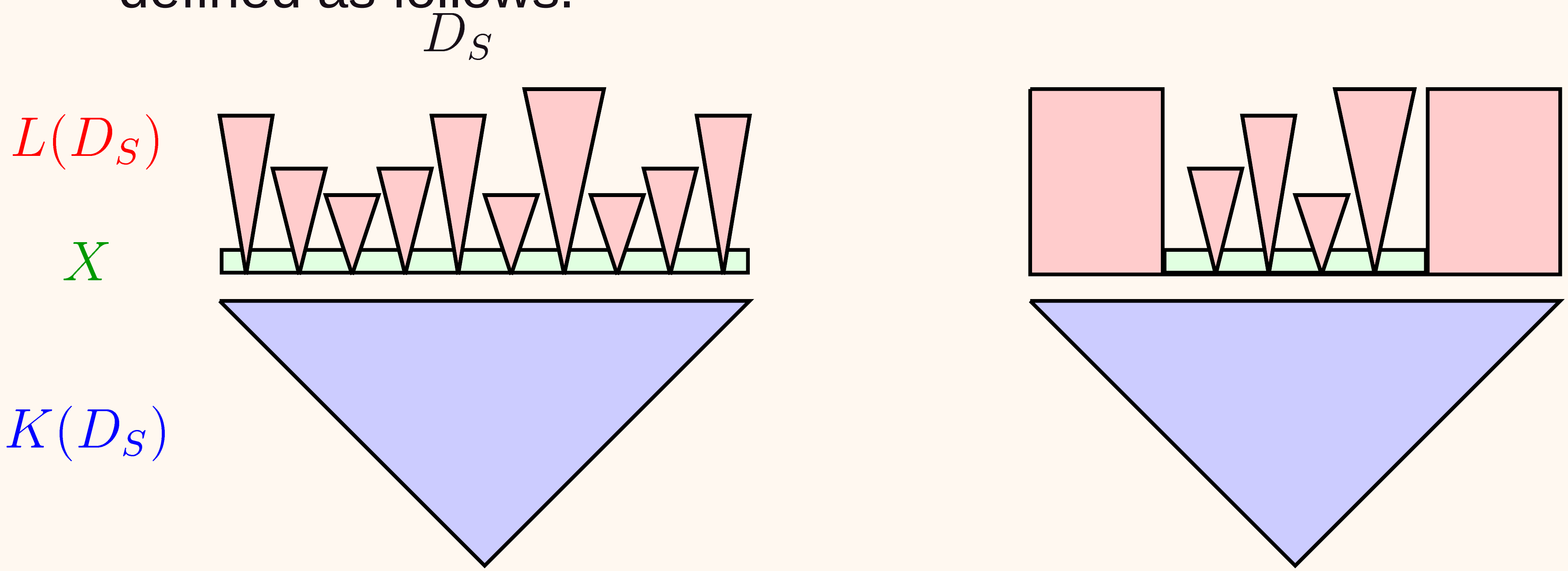
A sequence of covering induced by $K(D_S)$

- Suppose that X is regular (and thus metrizable), and S a proper dyadic subbase of X.
- We have a sequence $\mu_S = \mu_n (n = 0, 1, \ldots)$ of coverings defined as follows.



- Is it defining a uniformity on X?
  (It is, for the Gray-subbase of $\mathbb{I}$.)

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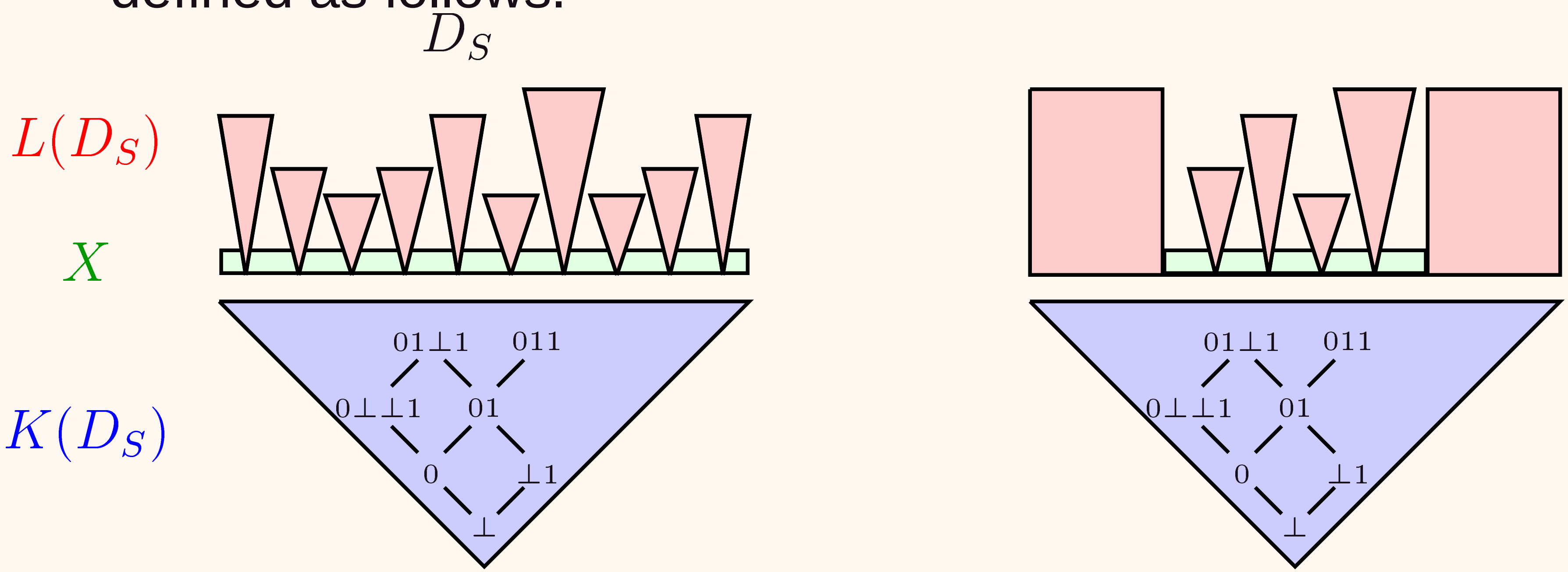

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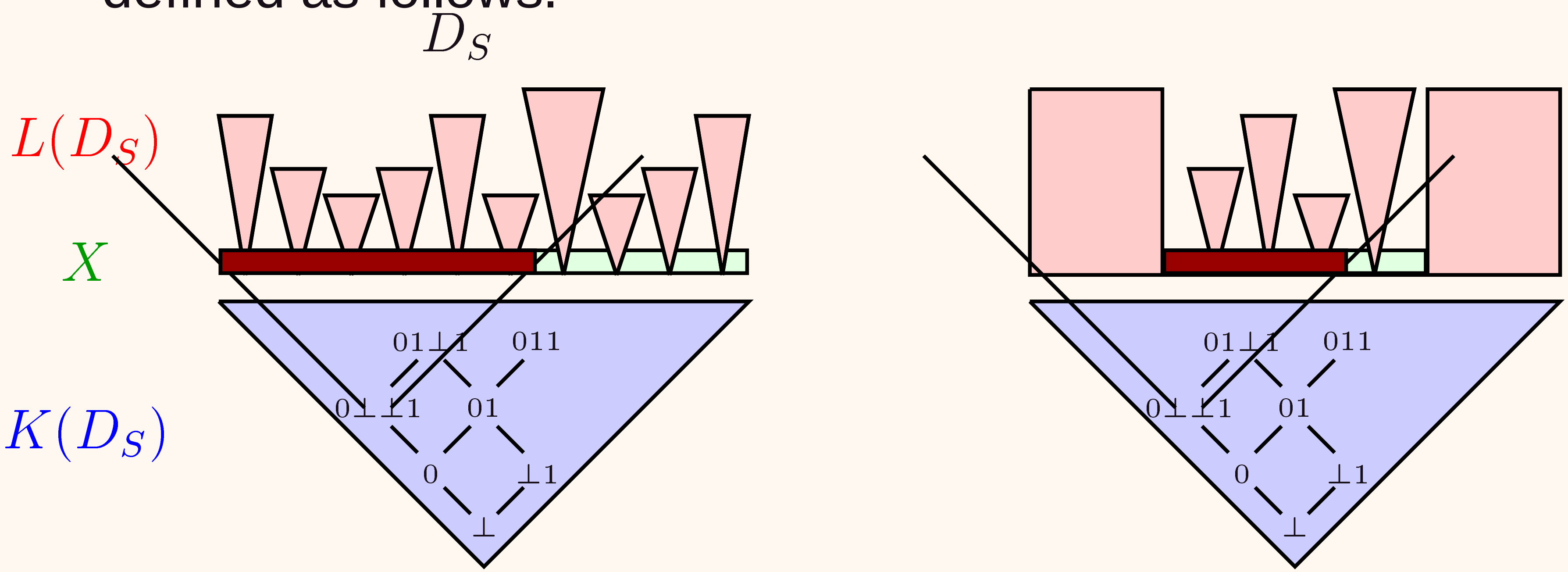

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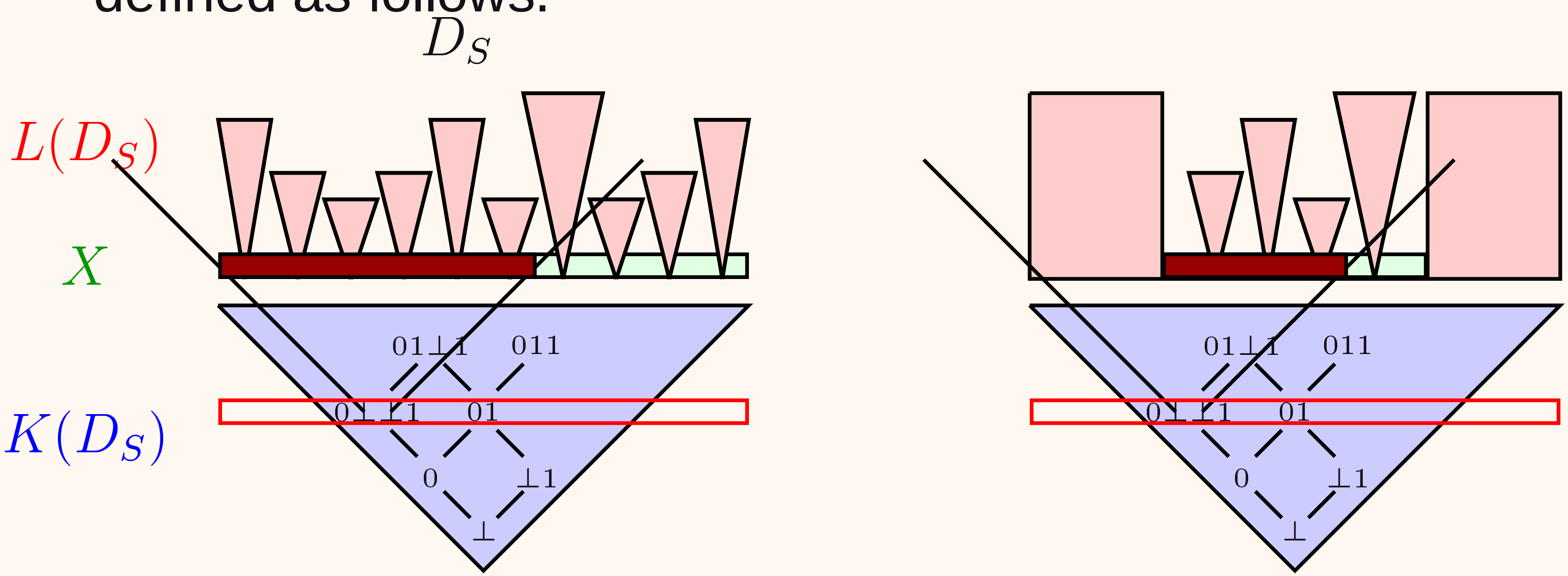

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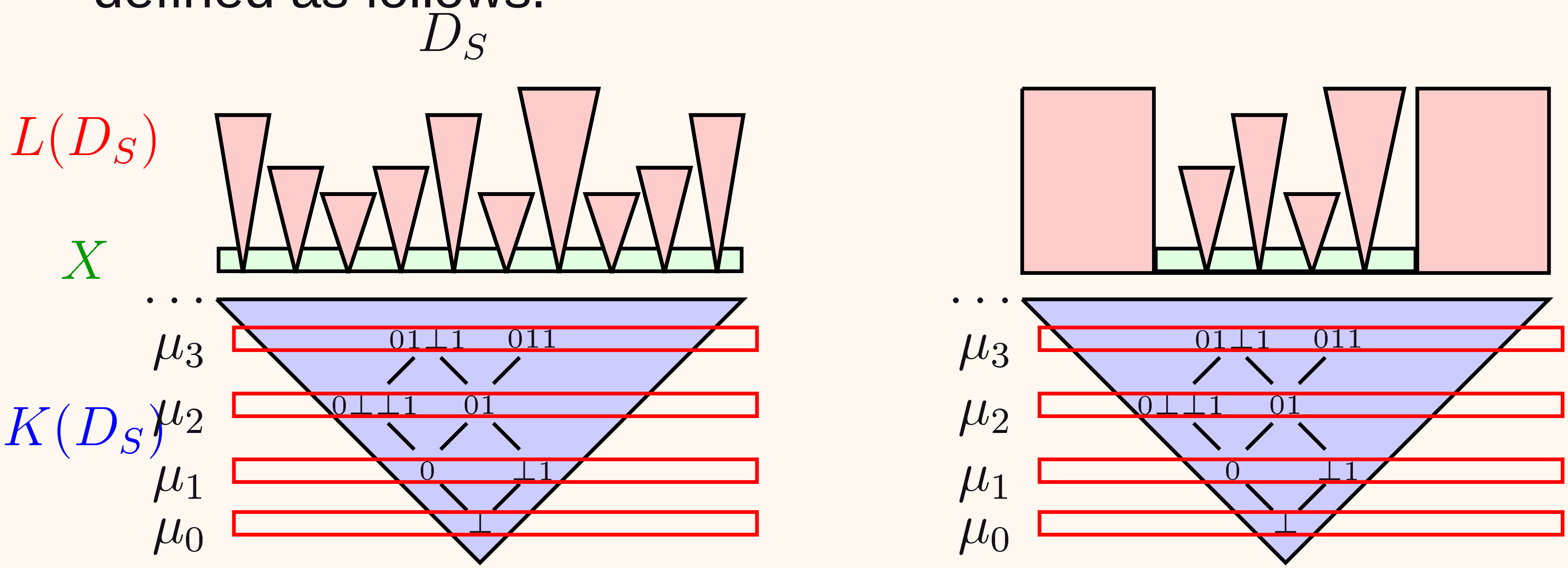

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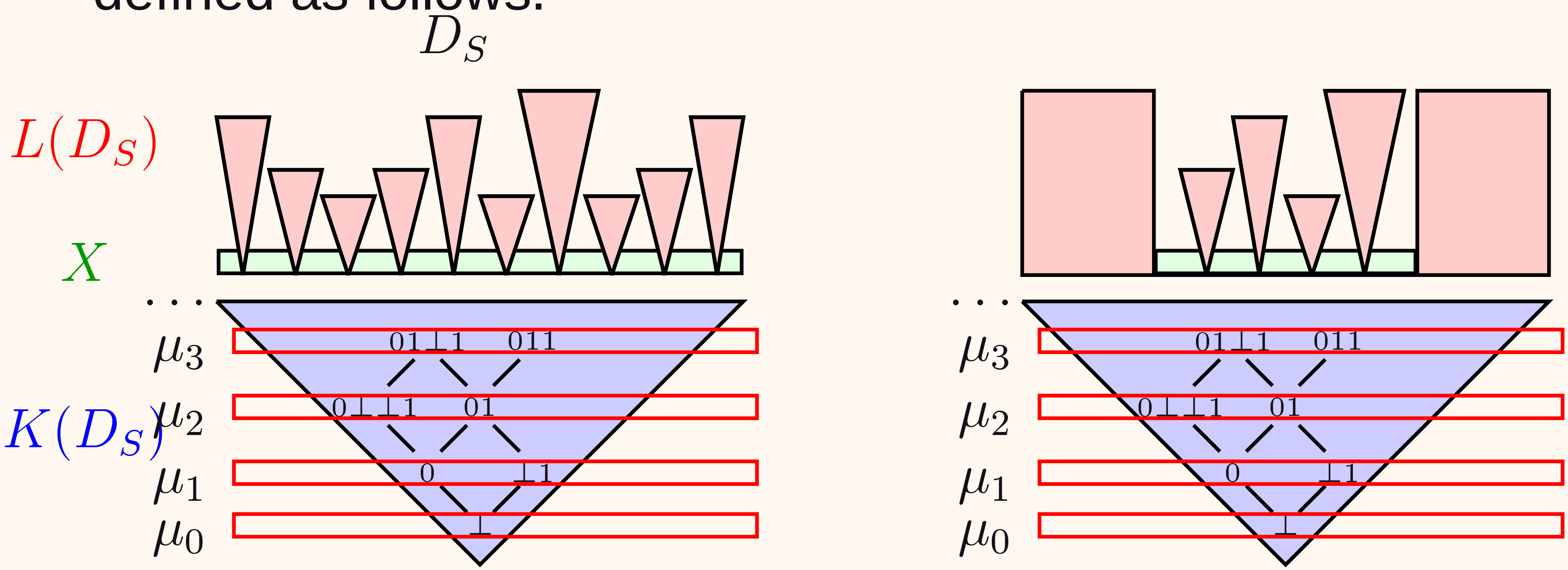

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Uniformity (via uniform coverings [Tukey40],[Isbell64])

Def. A family $\mathcal{U}$ of coverings of X is a *uniformity* if

(1) when μ and ν are in $\mathcal{U}$, $\mu \cap \nu$ is in $\mathcal{U}$,
(2) when $\mu \succ \nu$ and $\nu \in \mathcal{U}$, μ is in $\mathcal{U}$,
(3) every element of $\mathcal{U}$ has a star-refinement in $\mathcal{U}$, and
(4) for each x and $y \in X$, there is a covering $\mu \in \mathcal{U}$ no element of which contains both x and y.

- For a covering μ and $A \subset X$, $St(A, \mu) = \cup\{V \in \mu \mid V \cap A \neq \emptyset\}$.
- The collection $\{St(U, \mu) \mid U \in \mu\}$ is also a covering, called the star of μ and denoted by μ^*.
- If μ^* is a refinement of ν, we call that μ is a star-refinement of ν and write $\nu \succ^* \mu$.
- A sequence of covering $\mu_0 \succ \mu_1 \succ \mu_2 \succ \ldots$ is a countable base of a uniformity $\mathcal{U}$ if for all $\nu \in \mathcal{U}$, there is a n such that $\nu \succ^* \mu_n$.

Proper Dyadic subbase and uniformity

Theorem 7 *If X is a Compact Hausdorff Space, the sequence of covering μ_S is a base of the uniformity.*

For the case X is not compact, μ_S may not be a base of a uniformity, in general.

An example of an Adhesive Hausdorff Space

- D = the set of dyadic rationals of $[0, 1]$,
  $P = [0, 1] \setminus D$,
  $X = P \cup \mathbb{N}$.
- A neighbourhood base of $x \in P$: Euclidean neighbourhoods of x restricted to P.
- A neighbourhood base of $x \in \mathbb{N}$: Euclidean neighbourhoods of $\{k/2^x : k \text{ is odd}\}$ restricted to P extended with $\{x\}$.
- Every regular closed set contains $\{n \in \mathbb{N} : n \geq m\}$ for some $m \in \mathbb{N}$.