Geometric proof theory of intuitionistic geometry

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Brouwer's idea of 1924

- **apartness** $a \neq b$ of two real numbers as a basic notion
- properties, in logical notation:
  1. $\neg a \neq a$ *irreflexivity*
  2. $a \neq b \supset a \neq c \vee b \neq c$ *co-transitivity, "splitting"*

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- note the disjunction in a positive part
- equality defined: $a = b \equiv \neg a \neq b \ (\equiv a \neq b \supset \bot)$
- an "infinitely precise" notion

1. Heyting 1925

- the thesis:

Intuitionistische axiomatiek der projektieve meetkunde

2. Heyting 1927

- an easier read:

  Zur intuitionistischen Axiomatik der projektiven Geometrie
  (*Math. Ann.*, vol. 98, pp. 491–538)

3. Points in Heyting's intuitionistic geometry

- basic objects *points* that form the geometric space
- *apartness* and *equality* relations for points $A, B, C, \ldots$

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“Axiom I. Der Raum ist eine mathematische Spezies.”

“Axiom II. (*Axiom der Separation*.)

Wenn zwischen den Punkten A und B die Beziehung $A \, \omega B$ besteht, so gilt für jeden Punkt C entweder $A \, \omega C$ oder $B \, \omega C$.”

“Axiom III. Es können zwei voneinander entfernt liegende Punkte bestimmt werden.”

4. Lines and incidence

“Axiom IV. (Axiom der Geraden.) Die Geraden sind Punktspezies mit . . . ” (list of properties)

- a point is *incident* on a line if it belongs to the “species”

- *point outside a line* appears through a definition:

“Definition. Der Punkt *P* ist von der Punktspezies α entfernt (auch: liegt außerhalb der Punktspezies α), wenn er von jedem Punkt dieser Spezies entfernt ist; wir schreiben dann $P\omega\alpha$.”

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- I think this conceptual order of things was an error

5. Construction and existence

"Axiom III. Es können zwei voneinander entfernt liegende Punkte bestimmt werden."

"IVb. Zwei voneinander entfernt liegende Punkte bestimmen eine Gerade, die sie beide enthält (d. h. man kann eine Gerade l bestimmen, die sie beide enthält, und jede Gerade, die sie beide enthält, ist mit l identisch), ihre *Verbindunsgerade*."

"IVc. Jede Gerade enthält zum mindesten drei voneinander entfernt liegende Punkte."

"Axiom V. Außerhalb jeder Geraden kann ein Punkt bestimmt werden."

6. What to make of this?

- axiom IVb is clearly a construction postulate. An “infinitely precise” object is constructed that has some ideal, “infinitely precise” properties

- axioms III and V postulate “finitely precise” capacities

7. Some formal notation

- take *points* and *lines* as primitive, with an extra basic relation:

  $a \notin l$ *point a is outside line l* (JvP, APAL 1995)

- *incidence* defined:

  $a \in l \equiv \neg a \notin l$

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- now axiom III says: $\exists x \exists y . x \neq y$

- axiom V says: $\forall x \exists y . y \notin x$

(overloading of notation no problem: the sorts of objects are read off the relations)

8. The error in conceptual order

- Heyting's definition of "outside" is:

$$a \notin l \equiv \forall x(x \in l \supset a \neq x)$$

- the negative condition $x \in l$ is too weak to give $a \neq x$

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- compare

$$a \notin l \ \&\ a = b \supset b \notin l \qquad \text{(the standard substitution principle)}$$

1. Axioms of intuitionistic geometry

- *basic relations* $a \neq b, l \neq m, a \notin l$
- *connecting line* and *intersection point* constructions:

  $ln(a,b), pt(l,m)$, conditions $a \neq b, l \neq m$
- *incidence* properties:

  $a \in ln(a,b),\ b \in ln(a,b),\ pt(l,m) \in l,\ pt(l,m) \in m$
- *substitution* of equals:

  $a \notin l \supset a \neq b \vee b \notin l, \quad a \notin l \supset l \neq m \vee a \notin m$

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  $a \notin l \supset a \neq b \vee b \notin l, \quad a \notin l \supset l \neq m \vee a \notin m$
- note that these are non-Harrop formulas with essential disjunctions

2. Uniqueness

- the constructive axiom is:

$$a \neq b \,\&\, l \neq m \supset a \notin l \vee a \notin m \vee b \notin l \vee b \notin m$$

- classical contrapositive is *Skolem's axiom*:

$$a \in l \,\&\, a \in m \,\&\, b \in l \,\&\, b \in m \supset a = b \vee l = m$$

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- classical versions of substitutions are:

$$a \in l \,\&\, a = b \supset b \in l, \quad a \in l \,\&\, l = m \supset a \in m$$

- uniqueness the only disjunctive classical axiom

3. Non-collinearity

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- this is a *geometric implication* in the sense of category theory

4. From axioms to rules

- I turn the axioms into ND-style rules of the general form:

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$$\frac{\Gamma_1 \to \Delta_1, P_1 \qquad \dots \qquad \Gamma_m \to \Delta_m, P_m}{\Gamma_1, \dots, \Gamma_m \to \Delta_1, \dots, \Delta_m, Q_1, \dots, Q_n}$$

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- ND-style has the advantage: *Each conclusion is a premiss in the next rule, if there is one.*

- Sequent notation hopelessly messy in permutations, even unprintable

5. The geometric rules

I Rules for equality relations:

$$\frac{a \neq a}{\perp}\,Irref \qquad \frac{a \neq b}{b \neq a}\,Sym \qquad \frac{a \neq c}{a \neq b \quad b \neq c}\,Split$$

$$\frac{l \neq l}{\perp}\,Irref \qquad \frac{l \neq m}{m \neq l}\,Sym \qquad \frac{l \neq n}{l \neq m \quad m \neq n}\,Split$$

II Rules for incidence:

$$\frac{a \notin ln(a,b)}{\perp}\,ILn_1 \qquad \frac{b \notin ln(a,b)}{\perp}\,ILn_2$$

$$\frac{pt(l,m) \notin l}{\perp}\,IPt_1 \qquad \frac{pt(l,m) \notin m}{\perp}\,IPt_2$$

5. The geometric rules (cont.)

III Uniqueness rule:

$$\frac{a \neq b \quad l \neq m}{a \notin l \quad a \notin m \quad b \notin l \quad b \notin m}\,Uni$$

IV Substitution rules:

$$\frac{a \notin l}{a \neq b \quad b \notin l}\,SPt \qquad \frac{a \notin l}{l \neq m \quad a \notin m}\,SLn$$

5. The geometric rules (cont.)

V Rule of noncollinearity:

$$\dfrac{\begin{array}{c}[x \neq y, z \notin ln(x,y)] \\ \vdots \\ C\end{array}}{C}\,ET$$

- x, y, z eigenvariables, C arbitrary

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- Axioms and rules interderivable

- There are "too many rules" to analyze, but we can define point and line apartnesses and derive their six axioms:

$$a \neq b \equiv \exists x(a \notin x \,\&\, b \in x) \vee \exists y(a \in y \,\&\, b \notin x)$$

$$l \neq m \equiv \exists x(x \notin l \,\&\, x \in m) \vee \exists y(y \in l \,\&\, y \notin m)$$

6. Basic results

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Theorem. Word problem. *Derivability of $\Gamma \rightarrow \Delta$ by the rules of plane projective (resp. affine) intuitionistic geometry decidable, with provably terminating proof search.*

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Theorem. Word problem. *Derivability of $\Gamma \rightarrow \Delta$ by the rules of plane projective (resp. affine) intuitionistic geometry decidable, with provably terminating proof search.*

- The proof of lemma 1 is a monstrous seven-page tour of all the possibilities of combination of the 14 rules (five more pages for the 21 affine rules).

7. Outline of proof of lemma 1

1. Last occurrences of new terms. Note that all terms in premisses are found in some conclusion, and consider a new term in a loop-free derivation, say a line l. Trace it down. The following condition can be put:

Condition: The term l is a term of maximum length among all new terms in the derivation and the first in the lexicographical ordering of such terms. The downward branch is closed by an incidence rule, such as

$$\frac{a \notin ln(a,b)}{\perp}\,{\scriptstyle ILn_1} \qquad \frac{b \notin ln(a,b)}{\perp}\,{\scriptstyle ILn_2}$$

So $l \equiv ln(a,b)$

7. Outline of proof of lemma 1

2. Removal of new terms. Tracing up towards assumptions new terms from the roots, only *SLn* can make such a term $ln(a,b)$ appear, and $ln(a,b)$ occurs as a term in a line apartness in its conclusion. The only rule that can have such an apartness as a premiss in the derivation is *Uni* and we have, say,

$$\dfrac{c \neq d \qquad \dfrac{e \notin m}{m \neq ln(a,b) \qquad e \notin ln(a,b)}\,SLn}{c \notin m \quad c \notin ln(a,b) \quad d \notin m \quad d \notin ln(a,b)}\,Uni$$

7. Outline of proof of lemma 1

The proof continues by the following:

A. One instance of Uni. If the maximal number of instances of *Uni* with $In(a, b)$ in a premiss is 1 in the threads of a derivation, these instances can be converted so that no line apartness with the term $In(a, b)$ appears. Therefore there cannot be any new term $In(a, b)$ left.

B. Reduction of the number of Uni in the threads. If the maximal number of instances of *Uni* with $In(a, b)$ in a premiss is more than 1 in the threads of a derivation, it can be reduced.

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B. Reduction of the number of Uni in the threads. If the maximal number of instances of *Uni* with $ln(a, b)$ in a premiss is more than 1 in the threads of a derivation, it can be reduced.

Here is where the real work begins

8. Proof of lemma 2

Lemma 2 now follows easily: Consider an uppermost instance of rule *ET*. Its premiss is atomic and has been derived by rules I–IV, so the subterm property applies.
How can the terms x, y, z be removed from the cases of the derivation? Only by *Irref* or the incidence rules, but the former is excluded because $x \neq x$ and $y \neq y$ give loops (in rule *SPt*).
With *ILn*, the only line term with eigenvar's is $ln(x, y)$ so no instance of *ILn* can remove z distinct from x, y.

9. Affine geometry

The same results can be proved also for intuitionistic plane affine geometry.

Rules for constructed objects:

$$\frac{a \notin par(l, a)}{\perp}\,IA \qquad \frac{l \nparallel par(l, a)}{\perp}\,Par$$

Uniqueness of parallels:

$$\frac{l \neq m}{a \notin l \quad a \notin m \quad l \nparallel m}\,Unipar$$

Substitution rule:

$$\frac{l \nparallel m}{m \neq n \quad l \nparallel n}\,SA$$

10. A funny example

Euclid’s fifth postulate can be given as:

Given a point a outside line l, any point b is outside l or the parallel to l through point a.

$$a \notin l \rightarrow b \notin l, b \notin par(l, a)$$

Here is a derivation:

$$\dfrac{a \notin l}{\dfrac{a \notin par(l, a)}{\bot}\,IA \qquad \dfrac{l \neq par(l, a)}{b \notin l \quad b \notin par(l, a) \quad \dfrac{l \nparallel par(l, a)}{\bot}\,Par}\,Unipar}\,SLn$$

11. Syntactic proof of independence

Theorem. *If rule Unipar is deleted from the system of plane affine geometry and if the points a and b are not identical, the sequent*

$$a \notin l \rightarrow b \notin l, b \notin par(l, a)$$

is not derivable.

Proof. By the subterm property, only the four objects $a, b, l, par(l, a)$ need occur in proof search. Rule *SPt* applied to $a \notin l$ gives the cases $a \neq b, b \notin l$ the first of which is a dead end. Rule *SLn* gives the cases $a \notin par(l, a), l \neq par(l, a)$, and proof search ends. QED.

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Theorem. *The existence property fails in intuitionistic projective and affine geometry.*

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If $\exists xA$ derivable, then $A(t/x)$ derivable for some term t.

Theorem. *The existence property fails in intuitionistic projective and affine geometry.*

Proof: Assume the existence property. The sequent $\rightarrow \exists x \exists y \exists z(x \neq y \;\&\; z \notin In(x, y))$ is derivable, so $\rightarrow a \neq b \;\&\; c \notin In(a, b)$ derivable for some terms a, b, c. By the invertibility of the right conjunction rule in sequent calculus, $\rightarrow a \neq b$ and $\rightarrow c \notin In(a, b)$ are derivable. Proof search for these terminates without a derivation. QED.

13. Classical geometry

- equality and incidence as basic notions

- axioms classical contrapositives of the intuitionistic ones

- noncollinearity is:

$$\exists x \exists y \exists z(\neg x = y \,\&\, \neg z \in ln(x, y))$$

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- "Classical geometry is not a geometric theory" (Sara Negri, AML 2003)

14. Geometric and co-geometric axioms

- a formula is *geometric* if it does not contain $\supset$ or $\forall$.

- a *geometric implication* has the form, with A and B geometric formulas,

$$\forall x \ldots \forall z(A \supset B)$$

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- typical example:

$$\forall xyz (P_1 \& \ldots \& P_m \supset \exists uvw (Q_1 \& \ldots \& Q_n))$$

"For all x, y, z, if so-and-so, then there are u, v, w such that so-and-so."

- none of the "so-and-so's" can be conditionals or universals

- especially, no *negations*

15. The problem

- classical non-collinearity is not geometric:

$$\exists x \exists y \exists z(\neg x = y \,\&\, \neg z \in ln(x, y))$$

- intuitionistic non-collinearity is geometric:

$$\exists x \exists y \exists z(x \neq y \,\&\, z \notin ln(x, y))$$

15. The problem

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- intuitionistic non-collinearity is geometric:

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- The problem:

If we use existence axioms instead of constructions, these are geometric in classical geometry but non-collinearity is not geometric, and the other way around in intuitionistic geometry.

16. Co-geometric theories

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- classical projective and affine geometries with the axiom of non-collinearity are co-geometric

- the notion was invented by Sara Negri and yours truly on the basis of a proof-theoretical duality in rule systems

- do any “co”-results you please, ex. “co-Barr”

The point of intuitionistic geometry

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- alternative is to develop a theory first purely symbolically, and to postpone the interpretation of its basic objects and relations as much as possible

- good experiences with the implementation of intuitionistic theories of reals, and with intuitionistic geometry

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- our reasoning about them does not have this property

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