On sequential computability of a function

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Contents

1 Introduction

The universe of discourse:
Real numbers and real functions

Purpose:
A general treatment of *sequential computability* of some Euclidean-discontinuous functions

Sequential computable of f: $\{x_n\}$ computable $\to \{f(x_n)\}$ computable
Does not hold for a discontinuous function

A continuous function f is computable if f is sequentially computable and is effectively continuous

Two methods: Limiting recursion and effective uniformity

Our concern: Equivalence of sequential computabilities of the two methods under certain conditions for various examples

Preceding works: Equivalence proofs for concrete cases

Here: To distill a set of (sufficient) conditions under which the equivalence holds in a general setting

2 Preliminaries

A number-theoretic function is called ***limiting recursive*** (Gold) if it is recursive in the limits of recursive functions

(Sequence of real numbers)

A sequence of real numbers $\{x_m\}$ is E-*computable*/*weakly* E-*computable*:

It is approximated by a recursive (double) sequence of rational numbers (in the Euclidean topology)
with a recursive modulus of convergence/
with a limiting recursive modulus of convergence

(Effective uniformity on a set X)

$$U_n : X \to P(X)$$

$\mathcal{U} = \{U_n\}$ is an *effective uniformity* on X: It is a uniformity and the correspondences of indices are recursive.

There are recursive functions $\alpha_1, \alpha_2, \alpha_3$ which satisfy the following.

$$\forall x \in X. \cap_n U_n(x) = \{x\}$$
$$\forall n, m \forall x \in X. U_{\alpha_1(n,m)}(x) \subset U_n(x) \cap U_m(x)$$
$$\forall n \forall x, y \in X. x \in U_{\alpha_2(n)}(y) \to y \in U_n(x)$$
$$\forall n \forall x, y, z \in X. x \in U_{\alpha_3(n)}(y) \wedge y \in U_{\alpha_3(n)}(z) \to x \in U_n(z)$$

$\langle X, \{U_n\}\rangle$ is a (uniform) toplogical space with $\{U_n(x)\}$ as the system of fundamental neighborhoods.

$\langle X, \{U_n\}\rangle$: an *effective uniform space.*

Effective $\mathcal{U}$-convergence:

Convergence (of $\{r_{mk}\}$ to $\{x_m\}$) with respect to the topology $\mathcal{U}$ with a recursive modulus of convergence

$$\forall m \forall n \forall k \geq \gamma(m, n). r_{mk} \in U_n(x_m)$$

γ recursive

$\mathcal{U}$-computability structure $\mathcal{S} \subset P(X^\omega)$:

$\mathcal{S}$ is closed under recursive re-enumeration and $\mathcal{U}$-effective convergence.

A sequence in $\mathcal{S}$ is called $\mathcal{U}$-computable.

E-computable: Euclidean computable

$\mathcal{U} = \langle \mathrm{R}, \{U_n\}\rangle$: Effective uniform space on the domain of real numbers

Assumption $[\mathcal{A}]$ on $\mathcal{U}$:

$\mathcal{A}$-1 A recursive sequence of rational numbers is $\mathcal{U}$-computable.

$\mathcal{A}$-2 E-computable numbers and $\mathcal{U}$-computable numbers coincide.

$\mathcal{A}$-3 Every $\mathcal{U}$-computable sequence is E-computable.

Condition $[\mathcal{C}]$ on $\mathcal{U}$:

Any E-computable sequence $\{x_m\}$ is $\mathcal{U}$-approximated by a $\mathcal{U}$-computable sequence $\{z_{mp}\}$ and a limiting recursive modulus of convergence

($\{x_m\}$ is weakly $\mathcal{U}$-computable by $\{z_{mp}\}$ and ν)

$$\forall m, n \forall p \geq \nu(m,n). z_{mp} \in U_n(x_m)$$

Proposition 3.1 **If $\{x_m\}$ is $\mathcal{U}$-computable, then ν can be recursive, since $z_{mp} = x_m$ will do.**

Our Framework: $[\mathcal{A}]$ and $[\mathcal{C}]$.

4 Two notions of sequential computability of a function

($\mathcal{L}$-sequential computability)
A real function f is $\mathcal{L}$-*sequentially computable*:

For any E-computable sequence of real numbers $\{x_m\}$ (weakly $\mathcal{U}$-computable by $\{z_{mp}\}$ and ν by $[\mathcal{C}]$), $\{f(x_m)\}$ is weakly E-computable

(more precisely)
approximated by a recursive sequence of rational numbers $\{s_{mn}\}$ with a modulus of convergence η which is recursive in ν

Remark: In fact, $\mathcal{L}$-sequential computability of a function f should be defined independent of any uniformity. With each concrete example we have worked on, it was so defined, namely, a limiting modulus of convergence is defined independent of any uniformity, and later some conditions correspondig to those in the framework have been demonstrated. In an abstract setting, however, we must set those conditions as assumptions.

($\mathcal{U}$-sequential computability)
f **is** ***$\mathcal{U}$-sequentially computable*:**
For any $\mathcal{U}$-computable sequence of real numbers $\{x_m\}$, $\{f(x_m)\}$ is E-computable.

5 Mutual relationship of the two notions

Theorem 1. (From $\mathcal{L}$ to $\mathcal{U}$): If f is $\mathcal{L}$-sequentially computable, then f is $\mathcal{U}$-sequentially computable

Proof By the framework and Proposition 3.1

Note For Theorem 1, we do not need to assume any kind of continuity on the function f.

(Proof) of Theorem 1
Suppose f is $\mathcal{L}$-sequentially computable, and let $\{x_m\}$ be $\mathcal{U}$-computable. By [$\mathcal{A}$], $\{x_m\}$ is E-computable, and hence, by [$\mathcal{C}$], a limiting recursive ν is associated with it. So, by $\mathcal{L}$-sequential computability, there is a recursive sequence of rational numbers $\{t_{mp}\}$ and a function η which is recursive in ν satisfying

$$\forall m, p \forall q \geq \eta(m,p).|f(x_m) - t_{mq}| < \frac{1}{2^p}.$$

By virtue of Proposition 3.1, one can take a recursive ν for a $\mathcal{U}$-computable sequence $\{x_m\}$, and so we can take a recursive η so that $\{f(x_m)\}$ is E-computable by $\{t_{mq}\}$ and η, and hence f is $\mathcal{U}$-sequentially computable.

(Effective $\mathcal{U}$-continuity)
f is effectively $\mathcal{U}$-continuous:
There is a $\mathcal{U}$-computable sequence $\{e_i\}$ and a recursive function γ;

$$\forall p.\ \cup_i U_{\gamma(i,p)}(e_i) = \mathrm{R};$$

$$\forall p, i, x. x \in U_{\gamma(i,p)}(e_i)$$

$$\rightarrow |f(x) - f(e_i)| < \frac{1}{2^p}$$

Condition $[\mathcal{D}]$ on a function f:

$\mathcal{D}$-1: f is effectively $\mathcal{U}$-continuous with $\{e_i\}$ and γ

$\mathcal{D}$-2: For E-computable $\{x_m\}$ with ν, there is a function ι, recursive in ν,

$$\forall p, m. x_m \in U_{\gamma(\iota(m,p),p)}(e_{\iota(m,p)}).$$

$\mathcal{D}$-3: For E-computable $\{x_m\}$ with ν, there is a function ε, recursive in ν,

$$\forall i, m, n. x_m \in U_n(e_i)$$

$$\rightarrow U_{\varepsilon(m,i,n)}(x_m) \subset U_n(e_i)$$

Theorem 2. (From $\mathcal{U}$ to $\mathcal{L}$) Assume that [$\mathcal{D}$] holds for f. If f is $\mathcal{U}$-sequentially computable, then f is $\mathcal{L}$-sequentially computable

(Note) $\mathcal{D}$-1, that is, effective $\mathcal{U}$-continuity of f, is a natural condition, since the reason why we consider a uniformity is to make f continuous in the new topology so that the problem can be reduced to computability of a continuous function.

The conditions in $\mathcal{D}$-2 and $\mathcal{D}$-3 hold for the examples in preceding works (and in fact the corresponding functions are recursive: for example, $\varepsilon(m, i, n) = n$)

For some examples the theorem holds even without $\mathcal{U}$-continuity of f

Proof Suppose f is $\mathcal{U}$-sequentically computable.

Let $\{x_m\}$ be E-computable, and let p be a positive integer. We want to approximate $\{f(x_m)\}$ with a recursive rational (double) sequence with a modulus of conbergence recursive in ν

(i) By [$\mathcal{C}$], there are a $\mathcal{U}$-computable sequence $\{z_{mq}\}$ and a limiting recursive function ν such that

$$\forall m, n \forall q \geq \nu(m, n). z_{mq} \in U_n(x_m)$$

Since f is $\mathcal{U}$-sequentially computable and $\{z_{mp}\}$ is $\mathcal{U}$-computable, $\{f(z_{mp})\}$ is E-computable. From this follows:

(ii) There are a recursive sequence of rational numbers $\{s_{mql}\}$ and a recursive function β so taht

$$\forall n, m, q \forall l \geq \beta(m, q, n). |f(z_{mq}) - s_{mql}| < \frac{1}{2^n}$$

(iii) Put $t_{mq} := s_{mq\beta(m,q,q)}$.

(iv) By $\mathcal{D}$-2, there is an ι, recursive in ν, such that $x_m \in U_{\gamma(i,p+2)}(e_i)$ holds, where $i = \iota(m, p+2)$.

(v) From (iv) and $\mathcal{D}$-3, there is a function ε which is recursive in ν so that with $n = \gamma(i, p+2)$,

$$U_{\varepsilon(m,i,\gamma(i,p+2))}(x_m) \subset U_{\gamma(i,p+2)}(e_i).$$

For short, put $\varepsilon_0(m, i, p) = \varepsilon(m, i, \gamma(i, p+2))$.

(vi) In (i), put $n = \varepsilon_0(m, i, p)$ to obtain

$$q \geq \nu(m, \varepsilon_0(m, i, p)) \to z_{mq} \in U_{\varepsilon_0(m,i,p)}(x_m).$$

(vii) From (v) and (vi), we obtain

$$q \geq \nu(m, \varepsilon_0(m, i, p)) \to z_{mq} \in U_{\gamma(i,p+2)}(e_i).$$

(viii) From $\mathcal{D}$-1 with $x = z_{mq}$ and $p = p+2$, we obtain

$$z_{mq} \in U_{\gamma(i,p+2)}(e_i) \to |f(z_{mq}) - f(e_i)| < \frac{1}{2^{p+2}}.$$

(ix) From (vii) and (viii), follows

$$q \geq \nu(m, \varepsilon_0(m, i, p)) \to |f(z_{mq}) - f(e_i)| < \frac{1}{2^{p+2}}.$$

(x) In $\mathcal{D}$-1, put $x = x_m$ and $p = p + 2$. Since by (iv) $x_m \in U_{\gamma(i,p+2)}(e_i)$, it follows

$$|f(x_m) - f(e_i)| < \frac{1}{2^{p+2}}.$$

(xi) From (ii) and (iii) with $n = q$ and $l = \beta(m, q, q)$, we have

$$|f(z_{mq}) - s_{mq\beta(m,q,q)}| < \frac{1}{2^q}$$

or

$$|f(z_{mq}) - t_{mq}| < \frac{1}{2^q}.$$

(xii) Define

$$\delta(m, p) := \max(\nu(m, \varepsilon_0(m, i, p)), p + 2),$$

and notice that

$$q \geq \delta(m, p) \to q \geq p + 2 \to \frac{1}{2^q} \leq \frac{1}{2^{p+2}},$$

$$q \geq \delta(m, p) \to q \geq \nu(m, \varepsilon_0(m, i, p)).$$

(xiii) Summing up (ix)$\sim$(xii), we obtain, presuming that $q \geq \delta(m, p)$,

$$\begin{gathered} |f(x_m) - t_{mq}| \\ \leq |f(x_m) - f(e_i)| + |f(e_i) - f(z_{mq})| + |f(z_{mq}) - t_{mq}| \\ < \frac{3}{2^{p+2}} < \frac{1}{2^p}. \end{gathered}$$

$\{t_{mq}\}$ is a recursive sequence of rational numbers. Since $i = \iota(m, p)$ is recursive in ν, so is $\delta(m, p)$. (xiii) therefore proves that $\{f(x_m)\}$ is weakly E-computable by $\{t_{mq}\}$ and δ.

6 Examples

Example: the floor function $[x]$ and other piecewise continuous functions

Example: δ-function; heaviside function

Example: Fine continuous functions

Application of the gereral theory to the floor function

Let a be a recursive injection whose range is not recursive.

$$b_m = \begin{cases} 1 - \frac{1}{2^l} & \text{if } m = a(l) \text{ for some (unique) } l, \\ 1 & \text{otherwise.} \end{cases}$$

$\{b_m\}$ is E-computable, but $\{[b_m]\}$ is not. It is weakly E-computable.

$$\mathcal{U} = \langle \mathrm{R}, \{U_n\} \rangle:$$

$$U_n(x) = (x - \frac{1}{2^n}, x + \frac{1}{2^n}) \cap [l, l+1)$$

$$\text{if} \quad x \in [l, l+1), n = 0, 1, 2, \cdots$$

Effective uniformity

The floor function is $\mathcal{U}$-continuous

Computability structure $\mathcal{S}$:

recursive sequences of rational numbers;

the $\mathcal{U}$-effective limits of recursive sequences of rational numbers

$\{b_m\}$ is not $\mathcal{U}$-computable

[$\mathcal{A}$] and [$\mathcal{C}$] hold

E-computable sequence $\{x_m\} \to \{j_m\}$, recursive in ν,

$$x_m \in [j_m, j_m + 1)$$

Condition $\mathcal{D}$ hold for $[x]$

Proof $\mathcal{D}$-1: $\{e_i\}$ =the set of all integers

$\gamma(l,p) = 0$

Then $x \in U_{\gamma(l,p)}(l) = U_0(l) = [l, l+1)$ implies $[x] - [l] = 0$.

$\mathcal{D}$-2: E-computable $\{x_m\} \to$

$\iota(m,p) = j_m$

$\mathcal{D}$-3: Evaluate a $q = q_{m,n}$ such that $x_m + \frac{1}{2^q} < j_m + \frac{1}{2^n}$

Put $\varepsilon(m,i,n) = q_{mn}$ if $i = j_m$; $= 1$ otherwise

Fact: $[x]$ **is** $\mathcal{U}$- **sequentially computable (hence** $\mathcal{L}$**-sequentially computable)**

Proof If $\{x_m\}$ **is** $\mathcal{U}$**-computable with** $\{r_{mn}\}$ **and** α **and** $x_m \in [l, l+1)$**, then** $r_{m\alpha(m,0)} \in [l, l+1)$ **and hence** $[x_m] = [r_{m\alpha(m,0)}] = l$**. Since** $\{r_{m\alpha(m,0)}\}$ **is a recursive sequence of rational numbers,** $\{[r_{m\alpha(m,0)}]\}$ **is E-computable, hence so is** $\{[x_m]\}$**.**

Note $\mathcal{L}$**-sequential computability of** $[x]$ **has been directly shown in [YBW]**

(Limiting recursion method)
Advantage:

One attempts to compute the function values mechanically

The merit of this method: Simplicity

The only tool in need beyond the recursive function is taking the limit of a recursive function

A straight extension of the computation of continuous functions

Disadvantage: It does not seem to represent the mental activity of a mathematician (at least of mine) computing the function value at a point of discontinuity

(Effective uniformity method)

With each function a (an effective) uniform space in which it becomes continuous is associated

The theory of computability structure and computability of continuous functions can be applied

Except for recursive functions, we do not need any special tool beyond ordinary mathematical knowledge

Intuitive

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